Branches of non-symmetric critical points and symmetry breaking in nonlinear elliptic partial differential equations

Jean Dolbeault, Maria J. Esteban

Ceremade (UMR CNRS no. 7534), Univ. Paris-Dauphine, Pl. de Lattre de Tassigny, 75775 Paris Cedex 16, France

E-mail: `dolbeaul@ceremade.dauphine.fr, esteban@ceremade.dauphine.fr`

Abstract. In this paper we study the bifurcation of branches of non-symmetric solutions from the symmetric branch of solutions to the Euler-Lagrange equations satisfied by optimal functions in functional inequalities of Caffarelli-Kohn-Nirenberg type. We establish the asymptotic behavior of the branch for large values of the bifurcation parameter. We also perform a formal expansion in a neighborhood of the first bifurcation point on the branch of symmetric solutions, that characterizes the local behavior of the non-symmetric branch. These results are compatible with earlier numerical and theoretical observations. Further numerical results allow us to distinguish two global *scenarii*. This sheds a new light on the symmetry breaking phenomenon.

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1. Introduction

In this paper we investigate how symmetry can be broken in some variational problems when an external potential or a weight enters in competition with a nonlinearity. An archetypal example for such issues is the question of symmetry of optimal functions in Caffarelli-Kohn-Nirenberg inequalities. While all terms are invariant under rotation around the origin, it is known that optimizers are not always radially symmetric. Caffarelli-Kohn-Nirenberg inequalities, also known as Hardy-Sobolev inequalities, is a particularly simple setting for the study of symmetry breaking because weights and nonlinear terms have simple homogeneity properties, so that Euler-Lagrange equations inherit scaling properties that allow to further simplify the symmetry issues. Still, ranges

of parameters for which optimizers are radially symmetric have not been completely determined yet.

Symmetry breaking issues are present in many areas of physics involving partial differential equations: quantum mechanics, mean field models, equations for phase transition, ferromagnetism, mechanics, *etc.* Symmetry breaking occurs when antagonist effects are competing, like weights or potentials (or coupling with other fields) on the one hand and nonlinearites on the other hand. Various mathematical methods are available either for proving symmetry (uniqueness, comparison techniques based for instance on moving planes methods, symmetrization: see for instance [1, 2, 3]) or for proving symmetry breaking (multiplicity and bifurcation, energy, spectral methods). However, threshold cases are not characterized even in the simplest cases.

A simple mechanism which can break symmetry is the instability of the *symmetric extremals*, that is, the case where the extremals among radially symmetric functions are not local minima in the larger space of functions with no symmetry assumption. In the case of the Caffarelli-Kohn-Nirenberg inequalities, this instability has been studied in several papers (see [4, 5, 6]) and the corresponding region of symmetry breaking is delimited by a curve that we will call for brevity the *curve of Felli-Schneider*. However, it has been proved in [3] that symmetry breaking can occur even in a range of parameters for which the *symmetric extremals* are stable, that is, in cases where they are strict local minima. In order to understand this phenomenon, and symmetry breaking in general, we study the solution set associated to the Euler-Lagrange equations corresponding to a minimization problem associated to the Caffarelli-Kohn-Nirenberg inequalities. For those equations, we investigate the bifurcation of non-radially symmetric solutions from radially symmetric ones. The two theoretical contributions of the present paper are *an asymptotic analysis of the branches for large values of the bifurcation parameter*, in Section 2, and a detailed *expansion of the non-radial solutions in a neighborhood of the bifurcation point* on the branch of radial extremals, in Section 3. Both results are consistent with known numerical results and give a significant insight about the local behaviour of the solutions, either around the bifurcation point or asymptotically, that is numerically difficult to compute by direct methods. Let us give some further details.

We shall consider a subfamily of the Caffarelli-Kohn-Nirenberg inequalities which, for a given dimension $d \geq 3$, depend on two exponents, $p \in (2, 2^*]$ with $2^* := 2\,d/(d-2)$ and $\theta \in [\vartheta(p,d), 1]$ with $\vartheta(p,d) := d\,(p-2)/(2\,p)$, and on a parameter $\Lambda > 0$ (see Section 2.1). We shall also consider the two-dimensional case: $d = 2$, $p \in (2, 2^*)$ with $2^* = \infty$. Partial symmetry results are known from [7, 8, 9] and it has also been shown in [10, 3] that, in the set of parameters (p, Λ), a curve separates the region of symmetry from the region of symmetry breaking. *A priori* estimates can be used to localize this curve (primarily see [11], and references therein), but the curve itself has not been properly characterized yet. Our main purpose is to study branches of solutions to the Euler-Lagrange equations when $\theta = 1$ (see Section 2.1 below) and first check the conjecture

that the threshold is exactly given by the curve of Felli-Schneider, as was strongly suggested by [11]. We also aim at understanding the symmetry breaking observed in [3, Theorem 5] when $\theta < 1$. This can be done by reparametrizing the solutions set found in the case $\theta = 1$.

Numerical branches of solutions have been computed in great detail in [12]. The scheme allows us to construct branches of non-symmetric solutions which are candidates to be minimizers. Even if it cannot be granted that the computed solutions are extremals, their qualitative properties fully account for all observations, including the instability results in the range of parameters for which the symmetric extremals are stable. Based on these computations, two global *scenarii* can be distinguished at a qualitative level: for a given dimension $d \geq 2$ and a fixed exponent p, either the branch of non-symmetric solutions is monotone increasing for all admissible values of the parameter θ, or not. Numerically, we observe that the first *scenario* always holds for θ close to 1 while the second one is observed if p is not too large and only when θ is small enough. This picture is reinforced by two observations:

(1) What happens at the bifurcation point is difficult to characterize numerically using the scheme of [12]. However, the expansion of Section 3 shows that the branch is, locally monotone increasing if and only if θ is bigger than some threshold value.

(2) In the limit case $\theta = \vartheta(p,d)$, the branch of non-symmetric solutions has an asymptote. As in [13], the limiting energy can be compared with the energy of symmetric solutions at the bifurcation point. In case it is higher, we numerically observe that the branch is monotone increasing for all values of $\theta \in [\vartheta(p,d), 1]$, but not in the opposite case.

These two observations are consistent with the global qualitative *scenarii*. The numerical agreement of the threshold value for (1), which is local and restricted to a neighborhood of the bifurcation point, and the threshold value for (2), which is purely asymptotic, is striking. From a numerical point of view, this is our main result. This suggests that there is no other *scenario* than the two described above.

Our paper is organized as follows. Some known facts and preliminary estimates are collected in Section 2, including the method of the reparametrization of the branch of solutions for $\theta < 1$ by the solution set corresponding to $\theta = 1$. The analysis of the asymptotic regime as $\Lambda \to \infty$ is performed in Section 2.4 when $\theta \in [\vartheta(p,d), 1]$ and the corresponding result, Proposition 1, is new when $\theta \neq \vartheta(p,d)$ or 1. The expansion at the bifurcation point is done in Section 3, at a formal level. Indeed we make an *ansatz* for the non-symmetric solutions, and study the behavior of the branches near the bifurcation point within the corresponding class of functions. Most likely the *ansatz* can be justified to the price of complicated but elementary *a priori* estimates. This is the core of our contribution and it goes well beyond the stability analysis that can be found in [4, 5, 6]. Numerical results have been collected in Section 4 and discussed

there. Section 5 is devoted to some concluding remarks.

2. Preliminaries and theoretical results

2.1. Caffarelli-Kohn-Nirenberg inequalities

For any dimension $d \in \mathbb{N}^*$, let us consider the set $\mathcal{D}$ of all smooth functions which are compactly supported in $\mathbb{R}^d$. Define the numbers

$$\vartheta(p,d) := d\,\frac{p-2}{2\,p}\,,\;\; a_c := \frac{d-2}{2}\,,\;\; \Lambda(a) := (a-a_c)^2\,,\;\; p(a,b) := \frac{2\,d}{d-2+2\,(b-a)}\,.$$

We also set $2^* := \frac{2\,d}{d-2}$ if $d \geq 3$ and $2^* := \infty$ if $d = 1$ or 2. For any $a < a_c$, we consider the following *Caffarelli-Kohn-Nirenberg inequalities*, which have been introduced in [14] (also see [6]):

Let $b \in (a+1/2, a+1]$ and $\theta \in (1/2, 1]$ if $d = 1$, $b \in (a, a+1]$ if $d = 2$ and $b \in [a, a+1]$ if $d \geq 3$. Assume that $p = p(a,b)$, and $\theta \in [\vartheta(p,d), 1]$ if $d \geq 2$. Then, there exists a finite positive constant $\mathsf{K}_{\rm CKN}(\theta,\Lambda,p)$ such that, for any $w \in \mathcal{D}$,

$$\||x|^{-b}\,w\|^2_{\mathrm{L}^p(\mathbb{R}^d)} \leq \frac{\mathsf{K}_{\rm CKN}(\theta,\Lambda(a),p)}{|\mathbb{S}^{d-1}|^{\frac{p-2}{p}}}\,\||x|^{-a}\,\nabla w\|^{2\,\theta}_{\mathrm{L}^2(\mathbb{R}^d)}\,\||x|^{-(a+1)}\,w\|^{2\,(1-\theta)}_{\mathrm{L}^2(\mathbb{R}^d)}\,. \quad (1)$$

We denote by $\mathsf{K}^*_{\rm CKN}(\theta,\Lambda(a),p)$ the best constant among all radial functions. We recall that this constant is explicit (see [6, Lemma 3]) and equal to

$$\mathsf{K}^*_{\rm CKN}(\theta,\Lambda,p) = \mathsf{K}^*_{\rm CKN}(\theta,1,p)\,\Lambda^{-\frac{(2\,\theta-1)\,p+2}{2\,p}}\,,$$

with $\mathsf{K}^*_{\rm CKN}(\theta,1,p) = \left[\frac{(p-2)^2}{(2\,\theta-1)\,p+2}\right]^{\frac{p-2}{2\,p}} \left[\frac{(2\,\theta-1)\,p+2}{2\,p\,\theta}\right]^{\theta} \left[\frac{4}{p+2}\right]^{\frac{6-p}{2\,p}} \left[\frac{\Gamma\left(\frac{2}{p-2}+\frac12\right)}{\sqrt{\pi}\,\Gamma\left(\frac{2}{p-2}\right)}\right]^{\frac{p-2}{p}}.$

According to [5], the Caffarelli-Kohn-Nirenberg inequalities on $\mathbb{R}^d$ can be rewritten in cylindrical variables using the Emden-Fowler transformation

$$s = \log|x|\,,\quad \omega = \frac{x}{|x|} \in \mathbb{S}^{d-1}\,,\quad u(s,\omega) = |x|^{a_c-a}\,w(x)\,,$$

and are then equivalent to Gagliardo-Nirenberg-Sobolev inequalities on the cylinder $\mathcal{C} := \mathbb{R}\times\mathbb{S}^{d-1}$ that can be written as

$$\|u\|^2_{\mathrm{L}^p(\mathcal{C})} \leq \mathsf{K}_{\rm CKN}(\theta,\Lambda,p)\,\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathcal{C})} + \Lambda\,\|u\|^2_{\mathrm{L}^2(\mathcal{C})}\right)^{\theta}\,\|u\|^{2\,(1-\theta)}_{\mathrm{L}^2(\mathcal{C})} \quad (2)$$

for any $u \in H^1(\mathcal{C})$. *Here we adopt the convention that the measure on $\mathbb{S}^{d-1}$ is the uniform probability measure.*

Radial symmetry of some optimal functions in (1), or *symmetry* to make it short, means that there is an optimal function in (2) which only depends on s and $\mathsf{K}_{\mathrm{CKN}}(\theta,\Lambda,p) = \mathsf{K}^*_{\mathrm{CKN}}(\theta,\Lambda,p)$. We say that there is *symmetry breaking* if and only if $\mathsf{K}_{\mathrm{CKN}}(\theta,\Lambda,p) > \mathsf{K}^*_{\mathrm{CKN}}(\theta,\Lambda,p)$. Notice that on the cylinder the symmetric case is equivalent to the one-dimensional Gagliardo-Nirenberg-Sobolev inequality

$$\|u\|^2_{\mathrm{L}^p(\mathbb{R})} \leq \mathsf{K}^*_{\mathrm{CKN}}(\theta,\Lambda,p)\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R})} + \Lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{R})}\right)^{\theta}\|u\|^{2\,(1-\theta)}_{\mathrm{L}^2(\mathbb{R})}$$

for any $u \in H^1(\mathbb{R})$. Symmetry breaking of course makes sense only if $d \geq 2$ and we will assume it is the case from now on. Let us summarize known results. Let

$$\Lambda_{\mathrm{FS}}(p,\theta) := 4\,\frac{d-1}{p^2-4}\,\frac{(2\,\theta-1)\,p+2}{p+2} \quad \text{and} \quad \Lambda_\star(p) := \frac{(d-1)\,(6-p)}{4\,(p-2)}\,.$$

Symmetry breaking occurs for any $\Lambda > \Lambda_{\mathrm{FS}}(p,\theta)$ according to [4, 6] (also see [5] for previous results and [15] if $d = 2$ and $\theta = 1$). If $\theta = 1$, from [11], we know that symmetry holds for any $\Lambda \leq \Lambda_\star(p)$. Moreover, as shown in [15, 10, 3], there is a continuous curve $p \mapsto \Lambda_{\mathrm{s}}(p)$ with $\lim_{p\to 2_+}\Lambda_{\mathrm{s}}(p) = \infty$ and $\Lambda_{\mathrm{s}}(p) > a_c^2$ for any $p \in (2,2^*)$ such that symmetry holds for any $\Lambda \leq \Lambda_{\mathrm{s}}$ and there is symmetry breaking if $\Lambda > \Lambda_{\mathrm{s}}$. Despite the fact that the exact shape of Λ_s is not known, it can be proved that we have $\lim_{p\to 2^*}\Lambda_{\mathrm{s}}(p) = a_c^2$ if $d \geq 3$ and, if $d = 2$, $\lim_{p\to\infty}\Lambda_{\mathrm{s}}(p) = 0$, or more precisely, $\lim_{p\to\infty} p^2\Lambda_{\mathrm{s}}(p) = 4$. Moreover, we also know from [7, Theorem 3.1] that symmetry holds if $\Lambda \leq d^2/p^2$.

According to [16], existence of an optimal function is granted for any $\theta \in (\vartheta(p,d),1)$, but only if $\mathsf{K}_{\mathrm{CKN}}(\theta,\Lambda,p) > \mathsf{K}_{\mathrm{GN}}(p,d)$ when $\theta = \vartheta(p,d)$, where $\mathsf{K}_{\mathrm{GN}}(p,d)$ is the optimal constant in the following Gagliardo-Nirenberg-Sobolev inequality

$$\|u\|^2_{\mathrm{L}^p(\mathbb{R}^d)} \leq \frac{\mathsf{K}_{\mathrm{GN}}(p,d)}{|\mathbb{S}^{d-1}|^{(p-2)/p}}\,\|\nabla u\|^{2\,\vartheta(p,d)}_{\mathrm{L}^2(\mathbb{R}^d)}\,\|u\|^{2\,(1-\vartheta(p,d))}_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{R}^d)\,.$$

A sufficient condition for existence of extremals can be deduced, by comparison with symmetric functions, namely $\mathsf{K}^*_{\mathrm{CKN}}(\vartheta(p,d),\Lambda,p) > \mathsf{K}_{\mathrm{GN}}(p,d)$, which can be rephrased in terms of Λ as $\Lambda < \Lambda^*_{\mathrm{GN}}(p,d)$ for some non-explicit (but easy to compute numerically) function $p \mapsto \Lambda^*_{\mathrm{GN}}(p,d)$. When $\theta = \vartheta(p,d)$ and $\Lambda > \Lambda^*_{\mathrm{GN}}(p,d)$, extremal functions (if they exist) cannot be symmetric and in the asymptotic regime $p \to 2_+$, this condition is weaker than $\Lambda > \Lambda_{\mathrm{FS}}(p,\vartheta(p,d))$. One can indeed prove that $\lim_{p\to 2_+}\Lambda_{\mathrm{FS}}(p,\vartheta(p,d)) > \lim_{p\to 2_+}\Lambda^*_{\mathrm{GN}}(p,d)$. Hence, for $\theta \in (\vartheta(p,d),1]$, close enough to $\vartheta(p,d)$ and $p-2>0$, small (but numerically not so small, actually, as shown in [13]; also see [3, Section 5] for estimates and Section 4.3 for some plots), optimal functions exist and are not symmetric if $\Lambda > \Lambda^*_{\mathrm{GN}}(p,d)$, which is again a less restrictive condition than $\Lambda > \Lambda_{\mathrm{FS}}(p,\theta)$. See [3] for proofs and [13] for a more detailed overview of known results.

In this paper we study perturbatively the non-symmetric solutions lying in the first branch bifurcating from the branch of *symmetric extremals* and show that they

explain all phenomena of symmetry breaking known or observed so far, including cases in which the symmetric extremals are stable. Of course, it is not clear that all extremals for Caffarelli-Kohn-Nirenberg inequalities lie in those branches, even if probably that is the case. In this paper we provide a complete description of the branch around the bifurcation point, based on a formal asymptotic expansion. This clarifies the local behavior of the branch and accounts for all phenomena numerically observed in [12].

2.2. The case of symmetric extremals

We start with the symmetric case for $\theta = 1$ and adapt the computations that can be found in [6] (also see [11] and the Appendix). Consider the equation

$$-\,(p-2)^2\,w'' + 4\,w - 2\,p\,|w|^{p-2}\,w = 0 \quad \text{in} \quad \mathbb{R}\,. \tag{3}$$

The function $w(s) := (\cosh s)^{-\frac{2}{p-2}}$ is, up to translations, the unique positive solution of (3). As a consequence, the function $u(s) = \left(\frac{1}{2}\,p\,\mu\right)^{1/(p-2)}\,w\left(\frac{p-2}{2}\,\sqrt{\mu}\,s\right)$ is the unique solution of

$$-\;u'' + \mu\,u = |u|^{p-2}\,u \quad \text{in} \quad \mathbb{R}\,. \tag{4}$$

Now let us turn our attention to the case $\theta < 1$. Let

$$t[u] := \frac{\int_{\mathcal{C}} |\nabla u|^2\;dy}{\int_{\mathcal{C}} u^2\;dy}\,.$$

In the cylinder $\mathcal{C}$, solutions of the Euler-Lagrange equation satisfied by the extremals of (2) satisfy

$$-\;\theta\,\Delta u + \Big[(1-\theta)\,t[u] + \Lambda\Big]\,u = u^{p-1} \tag{5}$$

after a multiplication by an appropriate constant. Such a multiplication has indeed no incidence neither on the value of $t[u]$ nor on the value of the quotient

$$\mathcal{Q}^\theta_\Lambda[u] := \frac{\left(\int_{\mathcal{C}} |\nabla u|^2\;dy + \Lambda \int_{\mathcal{C}} u^2\;dy\right)^\theta\;\left(\int_{\mathcal{C}} u^2\;dy\right)^{1-\theta}}{\left(\int_{\mathcal{C}} u^p\;dy\right)^{\frac{2}{p}}}\,,$$

since $u \mapsto t[u]$ and $\mathcal{Q}^\theta_\Lambda[u]$ are both zero-homogeneous in u. Multiplying (5) by u and integrating on $\mathcal{C}$, we find that $\int_{\mathcal{C}} |\nabla u|^2\;dy + \Lambda \int_{\mathcal{C}} u^2\;dy = \int_{\mathcal{C}} u^p\;dy$. As a consequence, we have

$$\mathcal{Q}^\theta_\Lambda[u] = \|u\|^{\theta\,p-2}_{\mathrm{L}^p(\mathcal{C})}\;\|u\|^{2\,(1-\theta)}_{\mathrm{L}^2(\mathcal{C})}\,.$$

Denote by u_* the solution of (5) depending only on s. The same formulae as for u also hold for u_*. To relate $\mathsf{K}_{\mathrm{CKN}}(\theta,\Lambda,p)$ and $\mathsf{K}^*_{\mathrm{CKN}}(\theta,\Lambda,p)$ we have to compare $\mathcal{Q}^\theta_\Lambda[u]$

with $\mathcal{Q}^\theta_\Lambda[u_*]$. Then, either $\mathsf{K}_{\rm CKN}(\theta,\Lambda,p)=\mathsf{K}^*_{\rm CKN}(\theta,\Lambda,p)$ and symmetry holds, or the inequality

$$\mathcal{Q}^\theta_\Lambda[u]=\frac{1}{\mathsf{K}_{\rm CKN}(\theta,\Lambda,p)}\le\frac{1}{\mathsf{K}^*_{\rm CKN}(\theta,\Lambda,p)}=\mathcal{Q}^\theta_\Lambda[u_*]$$

is strict and then, there is symmetry breaking. The symmetric optimal function u_* for $\theta<1$ can be explicitly computed. Up to multiplication by a constant, u_* solves

$$-\,\theta\,u_*''+\eta\,u_*=u_*^{p-1}\,,$$

with $\eta=(1-\theta)\,t[u_*]+\Lambda$. After multiplying the above equation by u_*, integrating with respect to $s\in\mathbb{R}$ and dividing by $\int_{\mathbb{R}}u_*^2\,ds$, we find

$$t[u_*]+\Lambda=\frac{\int_{\mathbb{R}}u_*^p\,ds}{\int_{\mathbb{R}}u_*^2\,ds}$$

where $u_*(s)=A\,w(B\,s)$, for all $s\in\mathbb{R}$, w solves (3), $A=\left(\frac{p\,\eta}{2}\right)^{\frac{1}{p-2}}$ and $B=\frac{p-2}{2}\sqrt{\frac{\eta}{\theta}}$. From this expression, as in [11], we deduce that

$$t[u_*]=B^2\,\frac{\mathsf{J}_2}{\mathsf{I}_2}=\frac{p-2}{p+2}\,\frac{\eta}{\theta}\quad\text{and}\quad\frac{\int_{\mathbb{R}}u_*^p\,ds}{\int_{\mathbb{R}}u_*^2\,ds}=A^{p-2}\,\frac{\mathsf{I}_p}{\mathsf{I}_2}=\frac{2\,p\,\eta}{p+2}\,,$$

where for all $q\ge2$, $\mathsf{I}_q:=\int_{\mathbb{R}}|w(s)|^q\,ds$, and $\mathsf{J}_2:=\int_{\mathbb{R}}|w'(s)|^2\,ds$ (see Appendix A.1 for details). This provides the identity

$$\frac{p-2}{p+2}\,\frac{\eta}{\theta}+\Lambda=\frac{2\,p\,\eta}{p+2}$$

and uniquely determines $\eta=\frac{(p+2)\,\theta}{(2\,\theta-1)\,p+2}\,\Lambda$. As a consequence, we have

$$t[u_*]=\frac{p-2}{(2\,\theta-1)\,p+2}\,\Lambda\,.$$

2.3. Reparametrization of a branch corresponding to $\theta=1$ yielding a branch for $\theta<1$

We recall that $\theta\in[\vartheta(p,d),1]$ with $\vartheta(p,d)=d\,\frac{p-2}{2\,p}$. On the cylinder $\mathcal{C}:=\mathbb{R}\times\mathbb{S}^{d-1}$, we are interested in the minimization problem $1/\mathsf{K}_{\rm CKN}(\theta,\Lambda,p)=\inf_{v\in\mathrm{H}^1(\mathcal{C})\setminus\{0\}}\mathcal{Q}^\theta_\Lambda[v]$. Any minimizer v can be taken nonnegative without restriction and solves the Euler-Lagrange equation

$$-\,\Delta v+\frac{1}{\theta}\left[(1-\theta)\,t[v]+\Lambda\right]v-\frac{1}{\theta}\,\frac{\mathcal{Q}_\Lambda[v]}{\|v\|^{p-2}_{\mathrm{L}^p(\mathcal{C})}}\,v^{p-1}=0\quad\text{in}\quad\mathcal{C}\,,\tag{6}$$

with $t[v]=\frac{\|\nabla v\|^2_{\mathrm{L}^2(\mathcal{C})}}{\|v\|^2_{\mathrm{L}^2(\mathcal{C})}}$ and $\mathcal{Q}_\Lambda[v]:=\mathcal{Q}^{\theta=1}_\Lambda[v]=\frac{\|\nabla v\|^2_{\mathrm{L}^2(\mathcal{C})}+\Lambda\,\|v\|^2_{\mathrm{L}^2(\mathcal{C})}}{\|v\|^2_{\mathrm{L}^p(\mathcal{C})}}$.

If $\theta = 1$ and $\Lambda = \mu$, any nonnegative minimizer u solves the Euler-Lagrange equation

$$-\Delta u+\mu\,u-\frac{\mathcal{Q}_\mu[u]}{\|u\|_{\mathrm{L}^p(\mathcal{C})}^{p-2}}\,u^{p-1}=0\quad\text{in}\quad\mathcal{C}\,. \tag{7}$$

For any $\theta\in[\vartheta(p,d),1)$, solutions of (6) are in one-to-one correspondance with solutions of (7) through the relation $u=v$ and $(1-\theta)\,t[v]+\Lambda=\theta\,\mu$. Summarizing, if for any μ we have a solution u_μ of (7), and if we define

$$\tau(\mu):=t[u_\mu]\quad\text{and}\quad\nu(\mu):=\frac{\|u_\mu\|^2_{\mathrm{L}^2(\mathcal{C})}}{\|u_\mu\|^2_{\mathrm{L}^p(\mathcal{C})}}\,,$$

we can parametrize Λ and

$$J^\theta(\mu):=\mathcal{Q}^\theta_\Lambda[u_\mu]=\frac{\Big(\,\|\nabla u_\mu\|^2_{\mathrm{L}^2(\mathcal{C})}+\Lambda\,\|u_\mu\|^2_{\mathrm{L}^2(\mathcal{C})}\Big)^\theta\;\|u_\mu\|^{2\,(1-\theta)}_{\mathrm{L}^2(\mathcal{C})}}{\|u_\mu\|^2_{\mathrm{L}^p(\mathcal{C})}}$$

by

$$\Lambda^\theta(\mu)=\theta\,\mu-(1-\theta)\,\tau(\mu)\quad\text{and}\quad J^\theta(\mu)=\nu(\mu)\,\theta^\theta\,(\mu+\tau(\mu))^\theta\,. \tag{8}$$

For consistency, we adopt the notation u_μ for solutions of (7) which are non-symmetric in the range of symmetry breaking, and $u_{\mu,*}$ for the branch of symmetric solutions which are optimal in the corresponding class.

All computations of this section are valid for the branch of non-symmetric solutions, but can be done in the class of radial functions as well. With evident notations, we shall then denote by $u_{\mu,*}$ the solution of (4), and define $\tau_*(\mu)$, $\nu_*(\mu)$, $\Lambda^\theta_*(\mu)$ and $J^\theta_*(\mu)$

2.4. Gagliardo-Nirenberg inequalities and the corresponding asymptotic regime

In this section we investigate the asymptotic regimes corresponding to $\Lambda\to\infty$. Let

$$\mathsf{S}_p(\mathbb{R}^d):=\inf_{u\in\mathrm{H}^1(\mathbb{R}^d)\setminus\{0\}}\frac{\int_{\mathbb{R}^d}|\nabla u|^2\;dx+\int_{\mathbb{R}^d}|u|^2\;dx}{|\mathbb{S}^{d-1}|^{\frac{p-2}{p}}\;\left(\int_{\mathbb{R}^d}|u|^p\;dx\right)^{\frac2p}}\,,$$

and recall that the optimal constant $\mathsf{K}_{\mathrm{GN}}=\mathsf{K}_{\mathrm{GN}}(p,d)$ in the Gagliardo-Nirenberg-Sobolev inequality is given by

$$\frac{1}{\mathsf{K}_{\mathrm{GN}}}:=\inf_{u\in\mathrm{H}^1(\mathbb{R}^d)\setminus\{0\}}\frac{\|\nabla u\|^{2\,\vartheta}_{\mathrm{L}^2(\mathbb{R}^d)}\|u\|^{2\,(1-\vartheta)}_{\mathrm{L}^2(\mathbb{R}^d)}}{|\mathbb{S}^{d-1}|^{\frac{p-2}{p}}\;\|u\|^2_{\mathrm{L}^p(\mathbb{R}^d)}}\quad\text{with}\quad\vartheta=\vartheta(p,d)=d\,\frac{p-2}{2\,p}\,.$$

An optimization of the quotient in the expression of $\mathsf{S}_p(\mathbb{R}^d)$ allows to relate this constant with K_{GN}. Indeed, if we optimize $\mathcal{N}[u] := \int_{\mathbb{R}^d} |\nabla u|^2\,dx + \mu \int_{\mathbb{R}^d} |u|^2\,dx$ under the scaling $\lambda \mapsto u_\lambda(x) := \lambda^{\frac{d}{p}}\,u(\lambda\,x)$, we find that

$$\mathcal{N}[u_\lambda] = \lambda^{2\,(1-\vartheta)} \int_{\mathbb{R}^d} |\nabla u|^2\,dx + \lambda^{-2\,\vartheta}\,\mu \int_{\mathbb{R}^d} |u|^2\,dx$$

achieves its minimum at

$$\lambda_\star = \sqrt{\frac{\vartheta\,\mu}{1-\vartheta}}\,\frac{\|u\|_{\mathrm{L}^2(\mathbb{R}^d)}}{\|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^d)}}\,,$$

so that

$$\mathcal{N}[u_{\lambda_\star}] = \vartheta^{-\vartheta}\,(1-\vartheta)^{-(1-\vartheta)}\,\|\nabla u\|^{2\,\vartheta}_{\mathrm{L}^2(\mathbb{R}^d)}\|u\|^{2\,(1-\vartheta)}_{\mathrm{L}^2(\mathbb{R}^d)}\,\mu^{1-\vartheta}\,,$$

thus proving that, with the choice $\mu = 1$, $\mathsf{K}_{\mathrm{GN}}^{-1} = \vartheta^\vartheta\,(1-\vartheta)^{1-\vartheta}\,\mathsf{S}_p(\mathbb{R}^d)$. For any $\mu > 0$, if u_μ is the solution of (7) and if it is a minimizer of $1/\mathsf{K}_{\mathrm{CKN}}(1,\Lambda,p)$, we know from [5, Theorem 1.2] that as $\mu \to \infty$,

$$(\tau(\mu)+\mu)\,\nu(\mu) = \mathcal{Q}_\mu[u_\mu] \sim \mathsf{S}_p(\mathbb{R}^d)\,\mu^{1-\vartheta}\,.$$

If u is an optimal function for $\mathsf{S}_p(\mathbb{R}^d)$, we also know from the above computations that $\lambda_\star = 1$, that is,

$$1 = \lambda_\star^2 = \frac{\vartheta\,\mu}{1-\vartheta}\,\frac{1}{\tau(\mu)} \quad \text{and so}\,, \quad \frac{\tau(\mu)}{\mu} = \frac{\vartheta}{1-\vartheta}\,.$$

Hence,

$$\nu(\mu) \sim (1-\vartheta)\,\mathsf{S}_p(\mathbb{R}^d)\,\mu^{-\vartheta} \quad \text{as} \quad \mu \to \infty\,.$$

Consider now the case $\theta > \vartheta = \vartheta(p,d)$. According to (8), that is, by definition of J^θ and Λ^θ, we obtain that

$$\Lambda^\theta(\mu) = \theta\,\mu - (1-\theta)\,\tau(\mu) = \frac{\theta - \vartheta(p,d)}{1-\vartheta(p,d)}\,\mu\,,$$

$$J^\theta(\mu) = \nu(\mu)\,\theta^\theta\,(\mu + \tau(\mu))^\theta \sim \theta^\theta\,(1-\vartheta(p,d))^{1-\theta}\,\mathsf{S}_p(\mathbb{R}^d)\,\mu^{\theta-\vartheta(p,d)}\,,$$

as $\mu \to \infty$. Hence we have proved the following asymptotic result.

Proposition 1 *With the previous notations, for all $\theta > \vartheta = \vartheta(p,d)$,*

$$\lim_{\mu\to\infty} \mu^{\vartheta-\theta}\,J^\theta(\mu) = \frac{\theta^\theta}{\vartheta^\vartheta}\,(1-\vartheta)^{\vartheta-\theta}\,\mathsf{K}_{\mathrm{GN}}^{-1}\,.$$

Moreover, the parametric curve $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ *is asymptotic to the curve*

$$\Lambda \mapsto \frac{\theta^\theta \, (1-\vartheta(p,d))^{1-\vartheta(p,d)}}{(\theta-\vartheta(p,d))^{\theta-\vartheta(p,d)}} \, \mathsf{S}_p(\mathbb{R}^d) \, \Lambda^{\theta-\vartheta(p,d)} \,,$$

for large values of μ *or, equivalently, for large values of* $\Lambda = \Lambda^\theta(\mu)$.

See Figs. 1–3 for some plots of the curves $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ for various values of θ and how these curves can be compared with the ones corresponding to the asymptotic regime as described in Proposition 1.

The limit case $\theta = \vartheta = \vartheta(p,d)$ is of particular interest. Indeed, according to [16], Gagliardo-Nirenberg inequalities play a special role. See Fig. 6. First of all, since $1/\mathsf{K}_{\mathrm{CKN}}(\vartheta, \Lambda^\vartheta(\mu), p) \le J^\vartheta(\mu)$ and using the fact that $\Lambda \mapsto \mathsf{K}_{\mathrm{CKN}}(\vartheta, \Lambda, p)$ is a non-increasing function of Λ, we recover the known result that

$$\mathsf{K}_{\mathrm{GN}} \le \mathsf{K}_{\mathrm{CKN}}(\vartheta(p,d), \Lambda, p) \quad \forall\, \Lambda > 0 \,.$$

Such an inequality has deep implications on the existence of an optimal function (see [16] and in particular [16, Theorem 1.4]): either the inequality is strict and there exists a non-trivial optimal function for (1), or there is equality and a non-trivial optimal function may exist only if $\Lambda = \inf\{\lambda > 0 \,:\, \mathsf{K}_{\mathrm{GN}} \ge \mathsf{K}_{\mathrm{CKN}}(\vartheta(p,d), \lambda, p)\}$, but certainly not for any larger value of Λ, if the above infimum is finite.

In our setting, we can define $\mu_{\mathrm{GN}} := \inf\{\mu > 0 \,:\, J^\vartheta(\mu) \le \mathsf{K}_{\mathrm{GN}}\}$, with $\vartheta = \vartheta(p,d)$. It is granted that $\mu_{\mathrm{GN}} > 0$. Either $\mu_{\mathrm{GN}} = \infty$ and there is always a minimizer, or

$$\mathsf{K}_{\mathrm{CKN}}(\vartheta(p,d), \Lambda^\vartheta(\mu), p) = J^\vartheta(\mu) \quad \forall\, \mu \in (0, \mu_{\mathrm{GN}}]$$

and there exists a non-trivial optimal function for (1) if $\mu < \mu_{\mathrm{GN}}$, while

$$\mathsf{K}_{\mathrm{CKN}}(\vartheta(p,d), \Lambda^\vartheta(\mu), p) = \mathsf{K}_{\mathrm{GN}} \quad \forall\, \mu \in [\mu_{\mathrm{GN}}, \infty)$$

and there is no optimal function for (1) if $\mu > \mu_{\mathrm{GN}}$.

3. A formal expansion at the bifurcation point

In this section, we determine the behavior of the branch of non-symmetric positive solutions that bifurcates from the branch of the symmetric ones in a neighborhood of the first bifurcation point. Consider first the case $\theta = 1$ and denote by $u_{\mu,*}$ the positive symmetric solution of

$$-\,\Delta u + \mu\, u = u^{p-1} \,, \tag{9}$$

so that $\mathcal{Q}_\mu[u_{\mu,*}] = \|u_{\mu,*}\|_{\mathrm{L}^p(\mathcal{C})}^{p-2} = 1/\mathsf{K}^*_{\mathrm{CKN}}(1,\mu,p)$. Notice that if u is a solution to (9), we still have $\mathcal{Q}_\mu[u] = \|u\|_{\mathrm{L}^p(\mathcal{C})}^{p-2}$ even if u is not symmetric.

We will search for minimizers of $\mathcal{Q}_\mu$ in a restricted class of functions depending only on the variable s (see Section 2.1) along the axis of the cylinder and on the azimuthal angle ζ of the sphere because of the result on *Schwarz foliated symmetry* of [17]. This guarantees that we are in the right class for minimizers when $\theta = 1$. For $\theta < 1$, no such result has been established in the literature but we will work in the same framework. It is indeed straightforward to cheek that the same result holds.

3.1. Expansion of $\mathcal{Q}_\mu$ at order two

Let f_1 be the first non-constant spherical harmonic function, *i.e.* the eigenfunction of the Laplace-Beltrami operator on the sphere $\mathbb{S}^{d-1}$ corresponding to the eigenvalue $d-1$ and denote by f_2 the next one (among the ones depending only on the azimuthal angle ζ), with corresponding eigenvalue equal to $2\,d$. See Appendix A.4 for details. In the limiting regime corresponding to $\varepsilon \to 0$, an expansion at order two in ε gives

$$\frac{\mathcal{Q}_\mu[u_{\mu,*} + \varepsilon\,\varphi]}{\mathcal{Q}_\mu[u_{\mu,*}]} - 1 = \varepsilon^2\,\frac{q[\mu,\varphi]}{\|u_{\mu,*}\|_{\mathrm{L}^p(\mathcal{C})}^p} + \varepsilon^2\,(p-2)\,\frac{\left(\int_{\mathcal{C}} u_{\mu,*}^{p-1}\,\varphi\,dy\right)^2}{\|u_{\mu,*}\|_{\mathrm{L}^p(\mathcal{C})}^{2\,p}} + o(\varepsilon^2)\,,$$

where $q[\mu,\varphi] := \int_{\mathcal{C}} \left(|\nabla\varphi|^2 + \mu\,|\varphi|^2 - (p-1)\,u_{\mu,*}^{p-2}\,|\varphi|^2\right)\,dy$. By minimizing the term of order two, we find that

$$\frac{\mathcal{Q}_\mu[u_{\mu,*} + \varepsilon\,\varphi]}{\mathcal{Q}_\mu[u_{\mu,*}]} - 1 \sim \varepsilon^2\,\frac{(\varphi_1, \mathcal{H}_\mu\,\varphi_1)_{L^2(\mathcal{C})}}{\|u_{\mu,*}\|_{\mathrm{L}^p(\mathcal{C})}^p} \quad \text{as } \varepsilon \to 0\,,$$

where $\mathcal{H}_\mu := -\frac{d^2}{ds^2} + \mu + d - 1 - (p-1)\,u_{\mu,*}^{p-2}$ is a Pöschl-Teller operator whose lowest eigenvalue is given by $\lambda_1(\mu) = d - 1 + \mu - \frac{1}{4}\,\mu\,p^2$, and such that $\varphi = \varphi_1\,f_1$ is the corresponding eigenfunction (see Appendix A.2 for details). To fix notations, we normalize φ by the condition

$$\|\varphi\|_{\mathrm{L}^2(\mathcal{C})}^2 = \|\varphi_1\|_{\mathrm{L}^2(\mathbb{R})}^2 = \|u_{\mu,*}\|_{\mathrm{L}^p(\mathcal{C})}^p\,,$$

which slightly simplifies some computations below. This shows in particular that

$$\frac{q[\mu,\varphi]}{\|u_{\mu,*}\|_{\mathrm{L}^p(\mathcal{C})}^p} = \lambda_1(\mu)\,.$$

See (A.2) for an expression of φ_1.

As in [4], let μ_{FS} be such that $\lambda_1(\mu_{\mathrm{FS}}) = 0$, that is

$$\mu_{\mathrm{FS}} = 4\,\frac{d-1}{p^2-4}$$

(see Appendix A.2 for details). For any $\mu > \mu_{\rm FS}$, we have

$$\lambda_1(\mu) = -\frac{1}{4}\,(p^2-4)\,(\mu-\mu_{\rm FS})\,. \tag{10}$$

This determines the $O(\varepsilon^2)$ term. Now we want to investigate the behavior of $\mathcal{Q}_\mu$ in a neighborhood of $\mu = \mu_{\rm FS}$ and therefore need an expansion at higher order.

3.2. Expansion of $\mathcal{Q}_\mu$ at order four

Our purpose is to build a formal expansion of the branch of positive solutions of (9) that bifurcates from the branch $\mu \mapsto u_{\mu,*}$ at $u_{\mu_{\rm FS},*} =: u_{\rm FS}$ and satisfies $\mathcal{Q}_\mu[u_\mu] < \mathcal{Q}_\mu[u_{\mu,*}]$. For μ in a neighborhood of $\mu_{\rm FS}$, we look for solutions of (9) of the form

$$u_\mu = u_{\mu,*} + \varepsilon\,\varphi + \eta\,\psi + o(\varepsilon) + o(\eta)\,, \tag{11}$$

with $\varepsilon > 0$ and $\eta = o(\varepsilon)$. The fact that such an expansion can be built should be justified in a rigorous proof. We will not do it here and keep our computations at a formal level. Here $\varphi = \varphi_1\,f_1$ has been determined above. Recall that φ_1 is a function depending on s only. For convenience, let us define

$$k_\psi := \frac{\int_{\mathcal{C}} u_{\mu,*}^{p-1}\,\psi\;dy}{\int_{\mathcal{C}} u_{\mu,*}^{p}\;dy}\,.$$

Since we are interested in functions depending only on the azimuthal angle ζ, we indifferently use $\omega \in \mathbb{S}^{d-1}$ or $\zeta \in [0,\pi]$ with a slight abuse of notation. We consider the sequence $(f_k)_{k\in\mathbb{N}}$ of spherical harmonics depending only on ζ. See Appendix A.4 for details. We denote by ψ_k the decomposition of ψ in spherical harmonics:

$$\psi = k_\psi\,u_{\mu,*} + \sum_{k\ge 0} \psi_k\,f_k\,,$$

$$\text{with}\quad \psi_k(r) := \int_{\mathbb{S}^{d-1}} \psi(r,\omega)\,f_k(\omega)\,d\nu(\omega) \quad \forall\,r\in[0,\infty)\,,$$

where $d\nu(\omega)$ is the uniform probability measure on the sphere. Here we have chosen ψ_0 in such a way that $\int_{\mathcal{C}} u_{\mu,*}^{p-1}\,\psi_0\;dy = 0$ because

$$k_\psi = \frac{\int_{\mathcal{C}} u_{\mu,*}^{p-1}\,(k_\psi\,u_{\mu,*}+\psi_0)\;dy}{\int_{\mathcal{C}} u_{\mu,*}^{p}\;dy} = k_\psi + \frac{\int_{\mathcal{C}} u_{\mu,*}^{p-1}\,\psi_0\;dy}{\int_{\mathcal{C}} u_{\mu,*}^{p}\;dy}\,.$$

We know that $q[\mu,\psi] = \sum_{k\ge 0} q[\mu,\psi_k\,f_k]$, where

$$q[\mu,\psi_k\,f_k] = \int_{\mathcal{C}} |\nabla(\psi_k\,f_k)|^2\;dy + \mu\int_{\mathcal{C}} \psi_k^2\;dy - (p-1)\int_{\mathcal{C}} u_{\mu,*}^{p-2}\,\psi_k^2\;dy\,.$$

With $\mu_k := \mu + k\,(k+d-2)$, for any $k \geq 2$, we get that

$$q[\mu, \psi_k\, f_k] = \int_{\mathcal{C}} |\psi_k'|^2\, dy + \mu_k \int_{\mathcal{C}} \psi_k^2\, dy - (p-1) \int_{\mathcal{C}} u_{\mu,*}^{p-2}\, \psi_k^2\, dy$$

is nonnegative for $\mu - \mu_{\rm FS} > 0$, small enough, and positive unless $\psi_k \equiv 0$. Lengthy but straightforward computations show that, at least at formal level,

$$\frac{\mathcal{Q}_\mu[u_\mu]}{\mathcal{Q}_\mu[u_{\mu,*}]} - 1 = a\,\varepsilon^2 + b\,\varepsilon^4 + c\,\varepsilon^2\,\eta + d\,\eta^2 + e\,\varepsilon\,\eta + o(\varepsilon^4 + \eta^2 + |a|\,\varepsilon^2 + \eta\,\varepsilon^2)$$

(a rigorous justification of the order of the remainder terms requires some additional work), where

$$a(\mu) = \frac{q[\mu, \varphi]}{\int_{\mathcal{C}} u_{\mu,*}^p\, dy} = \lambda_1(\mu)\,,$$

$$\begin{aligned} \frac{b(\mu)}{p-1} = & -\lambda_1(\mu)\, \frac{\int_{\mathcal{C}} u_{\mu,*}^{p-2}\, \varphi^2\, dy}{\int_{\mathcal{C}} u_{\mu,*}^p\, dy} + \frac{1}{4}\,(p-1)\,(p-2) \left[\frac{\int_{\mathcal{C}} u_{\mu,*}^{p-2}\, \varphi^2\, dy}{\int_{\mathcal{C}} u_{\mu,*}^p\, dy} \right]^2 \\ & - \frac{1}{12}\,(p-2)\,(p-3)\, \frac{\int_{\mathcal{C}} u_{\mu,*}^{p-4}\, \varphi^4\, dy}{\int_{\mathcal{C}} u_{\mu,*}^p\, dy}\,, \end{aligned}$$

$$\begin{aligned} c(\mu) = & -2\,\lambda_1(\mu)\, k_\psi + (p-1)\,(p-2)\, \frac{\int_{\mathcal{C}} u_{\mu,*}^{p-2}\, \varphi^2\, dy}{\int_{\mathcal{C}} u_{\mu,*}^p\, dy}\, k_\psi \\ & - (p-1)\,(p-2)\, \frac{\int_{\mathcal{C}} u_{\mu,*}^{p-3}\, \varphi^2\, \psi\, dy}{\int_{\mathcal{C}} u_{\mu,*}^p\, dy}\,, \end{aligned}$$

$$d(\mu) = \frac{q[\mu, \psi]}{\int_{\mathcal{C}} u_{\mu,*}^p\, dy} + (p-2)\, k_\psi^2\,,$$

$$e(\mu) = 2\,\lambda_1(\mu)\, \frac{\int_{\mathcal{C}} \varphi\, \psi\, dy}{\int_{\mathcal{C}} u_{\mu,*}^p\, dy}\,.$$

With no restriction, we may require that φ is optimal in the direction f_1, that is

$$\int_{\mathcal{C}} \varphi\, \psi\, dy = 0\,. \tag{12}$$

Such an assumption is of course formal and a complete justification requires delicate estimates. This amounts to require that $e(\mu) = 0$ for any $\mu > \mu_{\rm FS}$.

According to (10), we get

$$a(\mu) = -\frac{1}{4}\,(p^2 - 4)\,(\mu - \mu_{\rm FS})\,.$$

Using $\int_0^\pi |f_1|^4\,d\nu = \frac{3\,d}{d+2}$ (see Appendix A.4), we obtain

$$\frac{4\,b(\mu_{\rm FS})}{(p-1)\,(p-2)} = (p-1)\left[\frac{\int_{\mathbb R} u_{\rm FS}^{p-2}\,\varphi_1^2\,ds}{\int_{\mathbb R} u_{\rm FS}^p\,ds}\right]^2 - \frac{d\,(p-3)}{d+2}\,\frac{\int_{\mathbb R} u_{\rm FS}^{p-4}\,\varphi_1^4\,ds}{\int_{\mathbb R} u_{\rm FS}^p\,ds}\,.$$

All above integrals are computed in Appendix A.3 and allow to express $b(\mu_{\rm FS})$ as

$$b(\mu_{\rm FS}) = \frac{4\,(d-1)^2\,p^3\,(p-1)^2\,[\,2\,p\,(5\,p-6) - d\,(p^2-16\,p+12)\,]}{(d+2)\,(p+2)^2\,(p-2)\,(3\,p-2)^2\,(5\,p-6)}\,.$$

As for the terms which depend on η, we observe that they sum as

$$\begin{aligned}
&\varepsilon^2\,\eta\,c(\mu) + \eta^2\,d(\mu) + \varepsilon\,\eta\,e(\mu)\\
&= \eta^2\left[\frac{q[\mu,\psi]}{\|u_{\mu,*}\|^p_{{\rm L}^p(\mathcal C)}} + (p-2)\,k_\psi^2\right] - 2\,\lambda_1(\mu)\,k_\psi\,\varepsilon^2\,\eta\\
&\quad + \varepsilon^2\,\eta\,(p-1)\,(p-2)\left[\frac{\int_{\mathcal C} u_{\mu,*}^{p-2}\,\varphi^2\,dy}{\|u_{\mu,*}\|^p_{{\rm L}^p(\mathcal C)}}\,k_\psi - \frac{\int_{\mathcal C} u_{\mu,*}^{p-3}\,\varphi^2\,\psi\,dy}{\|u_{\mu,*}\|^p_{{\rm L}^p(\mathcal C)}}\right]\,.
\end{aligned}$$

Using the fact that $f_1^2 = f_0 + \kappa_{(d)}\,f_2$ (see Appendix A.4), it is straightforward to observe that the minimizer is given by

$$\psi = k_\psi\,u_{\mu,*}\,f_0 + \psi_0\,f_0\ +\ \psi_1\,f_1\ +\ \psi_2\,f_2 \tag{13}$$

while $\psi_k \equiv 0$ for any $k>2$ and solves the Euler-Lagrange equation

$$\begin{aligned}
&-\Delta\psi + \mu\,\psi - (p-1)\,u_{\mu,*}^{p-2}\,\psi + (p-2)\,k_\psi\,u_{\mu,*}^{p-1}\,f_0\\
&\quad + \frac{\varepsilon^2}{2\,\eta}\left[\left(\frac{p^2\,(p-1)\,(p-2)}{3\,p-2}\,\mu - 2\,\lambda_1(\mu)\right)\,u_{\mu,*}^{p-1}\,f_0\right.\\
&\qquad\left. - (p-1)\,(p-2)\,u_{\mu,*}^{p-3}\,\varphi_1^2\,(f_0 + \kappa_{(d)}\,f_2)\right] + \mathcal L\,\varphi_1\,f_1 = 0\,.
\end{aligned}$$

Here we used the fact that $\int_{\mathcal C} u_{\mu,*}^{p-2}\,\varphi^2\ dy/\|u_{\mu,*}\|^p_{{\rm L}^p(\mathcal C)} = \frac{p^2\,\mu}{3\,p-2}$ (see Appendix A.3 for details). Constraint (12) is taken into account through the Lagrange multiplier $\mathcal L$, and a complete justification should take into account correction terms of higher order: see Remark 2.

The three components ψ_0, ψ_1 and ψ_2 satisfy the equations

$$\begin{aligned}
&-\psi_0'' + \mu\,\psi_0 - \frac{p\,(p-1)\,\mu}{2\,[\cosh(\beta\,s)]^2}\,\psi_0 + \frac{\beta^2\,A_0}{[\cosh(\beta\,s)]^{2\,\frac{p-1}{p-2}}} - \frac{\beta^2\,B_0}{[\cosh(\beta\,s)]^{2\,\frac{2\,p-3}{p-2}}} = 0\,,\\
&-\psi_1'' + \mu_1\,\psi_1 - \frac{p\,(p-1)\,\mu}{2\,[\cosh(\beta\,s)]^2}\,\psi_1 + \mathcal L\,\varphi_1 = 0\,,\\
&-\psi_2'' + \mu_2\,\psi_2 - \frac{p\,(p-1)\,\mu}{2\,[\cosh(\beta\,s)]^2}\,\psi_2 - \frac{\beta^2\,B_2}{[\cosh(\beta\,s)]^{2\,\frac{2\,p-3}{p-2}}} = 0\,,
\end{aligned}$$

with $\mu_1 = \mu + d - 1$, $\mu_2 = \mu + 2\,d$ and

$$A_0 = \frac{\alpha^{p-1}}{\beta^2}\,\frac{\varepsilon^2}{2\,\eta}\left(\frac{p^2\,(p-1)\,(p-2)}{3\,p-2}\,\mu - 2\,\lambda_1(\mu)\right)\,,$$
$$B_0 = \frac{\varepsilon^2}{2\,\eta}\,(p-1)\,(p-2)\,\frac{\alpha^{2\,p-3}}{\beta^2}\,,$$
$$B_2 = \frac{\varepsilon^2}{2\,\eta}\,(p-1)\,(p-2)\,\kappa_{(d)}\,\frac{\alpha^{2\,p-3}}{\beta^2} = \kappa_{(d)}\,B_0\,,$$

Recall that $\beta^2 = \frac{1}{4}\,(p-2)^2\,\mu$.

Multiplying the equation for ψ_1 by φ_1 and integrating by parts we get

$$\lambda_1(\mu)\int_{\mathbb{R}}\varphi_1\,\psi_1\,ds + \mathcal{L}\int_{\mathbb{R}}|\varphi_1|^2\,ds = 0\,.$$

Using Assumption (12), this proves that $\mathcal{L}=0$. This implies that ψ_1 is an eigenfunction of $\mathcal{H}_\mu$, with eigenvalue $\lambda_1(\mu)$. Since $\lambda_1(\mu)$ is simple, we find that $\psi_1\equiv 0$ by (12).

We may next observe that by taking

$$\psi_0(s) = A_0\,\chi_{0,p-1}(\beta\,s) + B_0\,\chi_{0,2\,p-3}(\beta\,s) \quad\text{and}\quad \psi_2(s) = B_2\,\chi_{2,2\,p-3}(\beta\,s)\,, \tag{14}$$

the problem is reduced to the set of equations

$$\begin{aligned}
&-\chi_{0,p-1}'' + \tfrac{4\,\chi_{0,p-1}}{(p-2)^2} - \tfrac{2\,p\,(p-1)\,\chi_{0,p-1}}{(p-2)^2\,(\cosh s)^2} + w^{p-1} = 0\,,\\
&-\chi_{0,2\,p-3}'' + \tfrac{4\,\chi_{0,2\,p-3}}{(p-2)^2} - \tfrac{2\,p\,(p-1)\,\chi_{0,2\,p-3}}{(p-2)^2\,(\cosh s)^2} - w^{2\,p-3} = 0\,,\\
&-\chi_{2,2\,p-3}'' + \tfrac{4\,\mu_2\,\chi_{2,2\,p-3}}{\mu\,(p-2)^2} - \tfrac{2\,p\,(p-1)\,\chi_{2,2\,p-3}}{(p-2)^2\,(\cosh s)^2} - w^{2\,p-3} = 0\,,
\end{aligned} \tag{15}$$

where $w(s) = (\cosh s)^{-\frac{2}{p-2}}$. Since

$$-\,w'' + \frac{4\,w}{(p-2)^2} - \frac{2\,p\,(p-1)\,w}{(p-2)^2\,(\cosh s)^2} + \frac{2\,p}{p-2}\,w^{p-1} = 0\,,$$

it follows that

$$\chi_{0,p-1} = \frac{p-2}{2\,p}\,w\,.$$

We may notice that the equations for $\chi_{0,p-1}$, $\chi_{0,2\,p-3}$ and $\chi_{2,2\,p-3}$ are all independent of k_ψ. Moreover, since $\int_{\mathbb{R}} u_{\mu,*}^{p-1}\,\psi_0\,ds = 0$, we get

$$\int_{\mathbb{R}} w^{p-1}\left(A_0\,\chi_{0,p-1} + B_0\,\chi_{0,2\,p-3}\right)ds = 0\,,$$

i.e.

$$0 = A_0\int_{\mathbb{R}}\frac{\chi_{0,p-1}}{(\cosh s)^{2\,\frac{p-1}{p-2}}}\,ds + B_0\int_{\mathbb{R}}\frac{\chi_{0,2\,p-3}}{(\cosh s)^{2\,\frac{p-1}{p-2}}}\,ds\,.$$

Remark 2 *The decomposition (13) of ψ is formal, because it is done up to higher order terms. Hence the above equality only holds for $\mu=\mu_{\rm FS}$, as can be checked by computing*

$$\int_{\mathbb{R}}\chi_{0,p-1}\,w^{p-1}\,ds=\frac{p-2}{2\,p}\int_{\mathbb{R}}w^p\,ds=\frac{p-2}{2\,p}\,\mathsf{I}_p$$

and, using (15),

$$\mathsf{b}_{0,p-1}:=\int_{\mathbb{R}}\chi_{0,2\,p-3}\,w^{p-1}\,ds=-\int_{\mathbb{R}}\chi_{0,p-1}\,w^{2\,p-3}\,ds=-\,\frac{p-2}{3\,p-2}\,\mathsf{I}_p\,.$$

(also see Appendix A.1). With $A_0=\frac{\alpha^{p-1}}{\beta^2}\,\frac{\varepsilon^2}{2\,\eta}\left(\frac{p^2\,(p-1)\,(p-2)}{3\,p-2}\,\mu-2\,\lambda_1(\mu)\right)$ and $B_0=\frac{\varepsilon^2}{2\,\eta}\,(p-1)\,(p-2)\,\frac{\alpha^{2\,p-3}}{\beta^2}$, we find that

$$A_0\,\frac{p-2}{2\,p}\,\mathsf{I}_p-B_0\,\frac{p-2}{3\,p-2}\,\mathsf{I}_p=0$$

holds if and only if $\lambda_1(\mu)=0$. As we shall see below, this is consistent with our expansion in terms of powers of ε and η because for $\mu>\mu_{\rm FS}$, close enough to $\mu_{\rm FS}$, $\lambda_1(\mu)$ corresponds to a term of higher order.

The reader is invited to check that

$$\chi_{0,2\,p-3}=-\,\frac{1}{4}\,\frac{p-2}{p-1}\left(2\,w-w^{p-1}\right)\,.$$

As a consequence, one can compute

$$\mathsf{b}_{0,1}:=\int_{\mathbb{R}}\chi_{0,2\,p-3}\,w\,ds=-\,\frac{1}{4}\,\frac{p-2}{p-1}\,(2\,\mathsf{I}_2-\mathsf{I}_p)=-\,\frac{p\,(p-2)}{2\,(p-1)\,(p+2)}\,\mathsf{I}_2\,.\qquad(16)$$

Altogether we have found that

$$\varepsilon^2\,\eta\,c(\mu)+\eta^2\,d(\mu)+\varepsilon\,\eta\,e(\mu)=\eta^2\,Q[\psi]-\varepsilon^2\,\eta\,L[\psi]$$

up to higher order terms in ε, η and $(\mu-\mu_{\rm FS})$, where

$$Q[\psi]:=\frac{q[\mu,\psi]}{\|u_{\mu,*}\|^p_{{\rm L}^p(\mathcal{C})}}+(p-2)\,k_\psi^2$$

and

$$L[\psi]:=-(p-1)\,(p-2)\left[\frac{\int_{\mathcal{C}}u_{\mu,*}^{p-2}\,\varphi^2\,dy}{\|u_{\mu,*}\|^p_{{\rm L}^p(\mathcal{C})}}\,k_\psi-\frac{\int_{\mathcal{C}}u_{\mu,*}^{p-3}\,\varphi^2\,\psi\,dy}{\|u_{\mu,*}\|^p_{{\rm L}^p(\mathcal{C})}}\right]$$

are respectively quadratic and linear with respect to ψ. Since we can multiply ψ by any positive constant ν and η by $1/\nu$ simultaneously without changing the value of $\eta^2\,Q[\psi]-\varepsilon^2\,\eta\,L[\psi]$, we do not introduce any restriction if we make the *ansatz*

$$\eta=\varepsilon^2\,, \tag{17}$$

thus making the sum of the two terms equal to $\varepsilon^4\,(Q[\psi]-L[\psi])$ and ψ independent of ε, at least at leading order. Moreover, if ψ is a minimizer of $Q[\psi]-L[\psi]$, then it is straightforward to check that $2\,Q[\psi]-L[\psi]=0$ (multiply the Euler-Lagrange equation by ψ and integrate). Altogether, we have found that

$$\varepsilon^2\,\eta\,c(\mu)+\eta^2\,d(\mu)+\varepsilon\,\eta\,e(\mu)=-\frac12\,\varepsilon^4\,L[\psi]<0$$

up to higher order terms in ε and for $\mu-\mu_{\rm FS}$ small enough, if ψ is a minimizer of $Q[\psi]-L[\psi]$, that is,

$$\frac{\mathcal Q_\mu[u_\mu]}{\mathcal Q_\mu[u_{\mu,*}]}-1-\lambda_1(\mu)\,\varepsilon^2-b(\mu)\,\varepsilon^4=-\frac12\,L[\psi]\,\varepsilon^4+o(\varepsilon^4)\,. \tag{18}$$

At this point, we may notice that the solution u_μ has not been normalized. Multiplying it by a constant would not change the value of $\mathcal Q_\mu[u_\mu]$. If we want it to be a solution of (9), then this implies that $\mathcal Q_\mu[u]=\|u\|^{p-2}_{\mathrm L^p(\mathcal C)}$ and we may therefore impose the corresponding constraint, *i.e.*

$$\int_{\mathcal C}|\nabla u_\mu|^2\,dy+\mu\int_{\mathcal C}u_\mu^2\,dy=\int_{\mathcal C}u_\mu^p\,dy\,,$$

without changing the equations written order by order (in other words, the Lagrange multiplier associated to this constraint is zero). Written in terms of φ and ψ, at lowest order, that is at order ε^2, this constraint amounts to

$$\int_{\mathcal C}\left(|\nabla\varphi|^2+\mu\,|\varphi|^2-\frac{p\,(p-1)}{2}\,u_{\mu,*}^{p-2}\,|\varphi|^2\right)\,dy-\,(p-2)\,k_\psi\int_{\mathcal C}u_{\mu,*}^p\,dy=0\,.$$

Hence by taking the limit as $\mu\to(\mu_{\rm FS})_+$ and observing that $\lambda_1(\mu)=O(\mu-\mu_{\rm FS})$, we find that, for $\mu=\mu_{\rm FS}$,

$$k_\psi=-\,\frac12\,(p-1)\,\frac{\int_{\mathcal C}u_{\mu,*}^{p-2}\,|\varphi|^2\,dy}{\int_{\mathcal C}u_{\mu,*}^p\,dy}=-\,\frac{2\,p^2\,(p-1)\,(d-1)}{(p-2)\,(p+2)\,(3\,p-2)}\,.$$

The explicit value of k_ψ will however not be needed later, because of cancellations that occur in all subsequent computations.

Next comes the observation that, as long as we are interested in computing $L[\psi]$, we do not even need to normalize u_μ nor to compute k_ψ. Indeed with $\widetilde\psi:=\psi-k_\psi\,u_{\mu,*}=$

$\psi_0\,f_0+\psi_2\,f_2$, we see that

$$\frac{\int_{\mathcal C} u_{\mu,*}^{p-2}\,\varphi^2\,dy}{\|u_{\mu,*}\|_{\mathrm L^p(\mathcal C)}^p}\,k_\psi-\frac{\int_{\mathcal C} u_{\mu,*}^{p-3}\,\varphi^2\,\psi\,dy}{\|u_{\mu,*}\|_{\mathrm L^p(\mathcal C)}^p}=-\frac{\int_{\mathcal C} u_{\mu,*}^{p-3}\,\varphi^2\,\widetilde\psi\,dy}{\|u_{\mu,*}\|_{\mathrm L^p(\mathcal C)}^p}$$

where $\widetilde\psi$ is fully determined by the coefficients A_0, B_0, and B_2, and Equation (15). This also determines

$$L[\psi]=(p-1)\,(p-2)\,\frac{\int_{\mathcal C} u_{\mu,*}^{p-3}\,\varphi^2\,\widetilde\psi\,dy}{\|u_{\mu,*}\|_{\mathrm L^p(\mathcal C)}^p}\,,$$

which is not known explicitly but is independent of k_ψ and can be computed as

$$\frac{\int_{\mathcal C} u_{\mu,*}^{p-3}\,\varphi^2\,\widetilde\psi\,dy}{\|u_{\mu,*}\|_{\mathrm L^p(\mathcal C)}^p}=\frac{A_0}{\alpha}\,\frac{p-2}{2\,p}\,\mathsf a_0+\frac{B_0}{\alpha}\,\alpha^{p-2}\,\frac{\mathsf b_{0,2\,p-3}}{\mathsf I_p}+\frac{B_2}{\alpha}\,\alpha^{p-2}\,\kappa_{(d)}\,\frac{\mathsf b_{2,2\,p-3}}{\mathsf I_p}$$

with

$$\mathsf a_0:=\frac{\int_{\mathcal C} u_{\mu,*}^{p-2}\,\varphi^2\,dy}{\int_{\mathcal C} u_{\mu,*}^{p}\,dy}=\frac{p^2\,\mu}{3\,p-2}\,,\quad \mathsf b_{0,2\,p-3}:=\int_{\mathbb R}\chi_{0,2\,p-3}\,w^{2\,p-3}\,ds\,,$$

$$\text{and}\quad \mathsf b_{2,2\,p-3}:=\int_{\mathbb R}\chi_{2,2\,p-3}\,w^{2\,p-3}\,ds\,.$$

Notice that here we have taken advantage of the facts that $\chi_{0,p-1}(\beta\,s)=\frac{p-2}{2\,p}\,\frac1\alpha\,u(s)$ and $\varphi^2=\varphi_1^2\,\big(f_0+\kappa_{(d)}\,f_2\big)$. Using the expression of $\chi_{0,2\,p-3}$, we can also compute

$$\begin{aligned}\mathsf b_{0,2\,p-3}&=-\,\frac14\,\frac{p-2}{p-1}\left(2\int_{\mathbb R}w^{2\,(p-1)}\,ds-\int_{\mathbb R}w^{3\,p-4}\,ds\right)\\&=-\,\frac{(p-2)\,p\,(3\,p-4)}{(p-1)\,(3\,p-2)\,(5\,p-6)}\,\mathsf I_p\,.\end{aligned}$$

Altogether, with $\mathsf y:=\frac{\mathsf b_{2,2\,p-3}}{\mathsf I_p}$, we have found that

$$L[\psi]=4\,(d-1)^2\,\frac{(p-1)\,p^3}{(p+2)^2}\left[\frac{(p-2)\,p}{(3\,p-2)^2\,(5\,p-6)}+2\,\frac{d-1}{d+2}\,\frac{p-1}{(p-2)^2}\,\mathsf y\right].$$

We have not been able to find an explicit expression for $\mathsf b_{2,2\,p-3}$, but we can prove that this is a positive quantity and even give an upper bound. According to [18, p. 74], the lowest eigenvalue of the Pöschl-Teller operator $-\frac{d^2}{ds^2}-U_0\,(\cosh s)^{-2}$ is given by

$$\lambda_0=\frac12\,\sqrt{1+4\,U_0}-\frac12-U_0\,,$$

if we assume that U_0 is positive. Here we have that U_0 is given by

$$U_0(p):=\frac{2\,p\,(p-1)}{(p-2)^2}$$

and the reader is invited to check that

$$\begin{aligned}\sigma(p,d) &:= \lambda_0 + \frac{4\,\mu_2}{\mu_{\rm FS}\,(p-2)^2}\\ &= \frac{1}{2}\sqrt{1+4\,U_0(p)} - \frac{1}{2} - U_0(p) + \frac{2}{(p-2)^2}\left(2 + \frac{d\,(p^2-4)}{d-1}\right)\end{aligned}$$

is larger than 1 for any $p > 2$ and any $d \ge 2$. As a straightforward consequence of (15), we deduce that

$$\sigma(p,d)\,\|\chi_{2,2\,p-3}\|^2_{\mathrm{L}^2(\mathbb{R})} \le \int_{\mathbb{R}} \chi_{2,2\,p-3}\,w^{2\,p-3}\,ds \le \|\chi_{2,2\,p-3}\|_{\mathrm{L}^2(\mathbb{R})}\,\|w^{2\,p-3}\|_{\mathrm{L}^2(\mathbb{R})}$$

and, finally,

$$\mathsf{b}_{2,2\,p-3} \le \frac{1}{\sigma(p,d)}\int_{\mathbb{R}} w^{2\,(2\,p-3)}\,ds = \frac{16\,p\,(p-1)\,(3\,p-4)}{(3\,p-2)\,(5\,p-6)\,(7\,p-10)}\,\frac{\mathsf{I}_p}{\sigma(p,d)}\,.$$

3.3. Optimization

Collecting the above estimates and using (18), we get

$$\frac{\mathcal{Q}_\mu[u_\mu]}{\mathcal{Q}_\mu[u_{\mu,*}]} = 1 - \frac{1}{4}\,(p^2-4)\,(\mu-\mu_{\rm FS})\,\varepsilon^2 + \left[b(\mu) - \frac{1}{2}\,L[\psi]\right]\varepsilon^4 + o(\varepsilon^4)\,,$$

provided $\mu - \mu_{\rm FS} = O(\varepsilon^2)$. Let

$$c_{p,d} := \frac{1}{8}\,(p^2-4)\left[b(\mu) - \frac{1}{2}\,L[\psi]\right]^{-1}_{|\mu=\mu_{\rm FS}}\,,$$

so that, in a neighborhood of $\mu = \mu_{\rm FS}$,

$$\frac{\mathcal{Q}_\mu[u_\mu]}{\mathcal{Q}_\mu[u_{\mu,*}]} = 1 - \frac{1}{4}\,(p^2-4)\left((\mu-\mu_{\rm FS})\,\varepsilon^2 - \frac{\varepsilon^4}{2\,c_{p,d}}\right) + o(\varepsilon^4)$$

is optimized, up to higher order terms, by taking

$$\varepsilon^2 = \varepsilon^2(\mu) \sim c_{p,d}\,(\mu-\mu_{\rm FS}) \quad \text{as} \quad \mu \to \mu_{\rm FS+} \tag{19}$$

if $c_{p,d}$ is positive. Altogether, we have found that

$$\frac{\mathcal{Q}_\mu[u_\mu]}{\mathcal{Q}_\mu[u_{\mu,*}]} = 1 - \frac{p^2-4}{8}\,c_{p,d}\,(\mu-\mu_{\rm FS})^2 + o\left((\mu-\mu_{\rm FS})^2\right)\,. \tag{20}$$

3.4. A consistency check

All above computations are valid under the assumption that $c_{p,d}$ *is positive*, but this is not *a priori* granted. With the estimate of $\mathsf{b}_{2,2\,p-3}$ that has been found at the end of Section 3.2, we can *a posteriori* give a sufficient condition for the consistency of the method. Since

$$\begin{aligned} L[\psi] &\le L_{\mathrm{approx}}[\psi] \\ &:= 4\,(d-1)^2\,\frac{(p-1)\,p^3}{(p+2)^2}\left[\frac{(p-2)\,p}{(3\,p-2)^2\,(5\,p-6)}+2\,\frac{d-1}{d+2}\,\frac{p-1}{(p-2)^2}\,\mathsf{y}\right]. \end{aligned}$$

with $\mathsf{y} := \frac{16\,p\,(p-1)\,(3\,p-4)}{(3\,p-2)\,(5\,p-6)\,(7\,p-10)\,\sigma(p,d)}$, we know that $c_{p,d}$ is well defined and positive if $L_{\mathrm{approx}}[\psi] < 2\,b(\mu_{\mathrm{FS}})$. Moreover, we have

$$c_{p,d} \le c_{p,d}^{\mathrm{approx}} := \frac{1}{8}\,(p^2-4)\,\Big[b(\mu)-\frac{1}{2}\,L_{\mathrm{approx}}[\psi]\Big]^{-1}_{|\mu=\mu_{\mathrm{FS}}}\,,$$

at least as long as $L_{\mathrm{approx}}[\psi] \le 2\,b(\mu_{\mathrm{FS}})$. This last condition holds if p is contained in a non empty interval $(2, p_{\mathrm{approx}}) \subset (2, 2^*)$, where p_{approx} is determined as the largest root of a fourth order polynomial, namely $p \mapsto \frac{1}{5}\,(54-227\,d+103\,d^2)\,p^4-4\,(18-37\,d+25\,d^2)\,p^3+\frac{8}{3}\,(63-67\,d+46\,d^2)\,p^2+16\,(d+3)\,(5\,d-3)\,p-240\,d\,(d+1)$. In practice $d \mapsto p_{\mathrm{approx}}(d)$ is close to $2\,d/(d-2)$ and converges to 2 as $d \to \infty$. See Fig. 11.

3.5. Expansion of $\tau(\mu)$ around μ_{FS}

Let $\tau(\mu) = t[u_\mu] = \frac{\int_{\mathcal{C}} |\nabla u_\mu|^2\,dy}{\int_{\mathcal{C}} u_\mu^2\,dy}$ and

$$\tau_*(\mu) := t[u_{\mu,*}] = \frac{p-2}{p+2}\,\mu\,. \tag{21}$$

We can notice that $\tau(\mu_{\mathrm{FS}}) = \beta^2\,\mathsf{J}_2/\mathsf{I}_2$ (see Appendix A.1 and Appendix A.2) so that

$$\tau(\mu_{\mathrm{FS}}) = \frac{p-2}{p+2}\,\mu_{\mathrm{FS}} = \frac{4\,(d-1)}{(p+2)^2}\,.$$

With the above expressions in hand, we can now compute the derivative

$$\tau'(\mu_{\mathrm{FS}}) = \frac{d}{d\mu}\,t[u_\mu]_{|\mu=\mu_{\mathrm{FS}}}\,.$$

From (21) we know that $t[u_{\mu,*}] - t[u_{\mathrm{FS}}] = \frac{p-2}{p+2}\,(\mu-\mu_{\mathrm{FS}})$. By expanding the expression $\tau(\mu)-\tau(\mu_{\mathrm{FS}}) = t[u_\mu]-t[u_{\mathrm{FS}}] = t[u_{\mu,*}]-t[u_{\mathrm{FS}}]+t[u_\mu]-t[u_{\mu,*}]$ in powers of ε with $u_\mu = u_{\mu,*}+\varepsilon\,\varphi_1\,f_1+\varepsilon^2\psi+o(\varepsilon^2)$ where φ, ψ have been chosen in Section 3.2, we get, up

to higher order terms,

$$t[u_\mu]-t[u_{\mu,*}]\sim\varepsilon^2\Bigg[(\lambda_1(\mu)-\mu-t[u_{\mu,*}])\frac{\int_{\mathcal C}\varphi^2\,dy}{\int_{\mathcal C}u_{\mu,*}^2\,dy}+(p-1)\,\frac{\int_{\mathcal C}u_{\mu,*}^{p-2}\,\varphi^2\,dy}{\int_{\mathcal C}u_{\mu,*}^2\,dy}$$
$$-2\,(\mu+t[u_{\mu,*}])\,\frac{\int_{\mathcal C}u_{\mu,*}\,\psi\,dy}{\int_{\mathcal C}u_{\mu,*}^2\,dy}+2\,\frac{\int_{\mathcal C}u_{\mu,*}^{p-1}\,\psi\,dy}{\int_{\mathcal C}u_{\mu,*}^2\,dy}\Bigg]\,,$$

and, by computing as above, we find that

$$\tau'(\mu_{\rm FS})=\frac{p-2}{p+2}+c_{p,d}\left[-\frac{2\,p\,\mu_{\rm FS}}{p+2}\,\frac{\int_{\mathcal C}\varphi_1^2\,dy}{\int_{\mathcal C}u_{\rm FS}^2\,dy}+(p-1)\,\frac{\int_{\mathcal C}u_{\rm FS}^{p-2}\,\varphi_1^2\,dy}{\int_{\mathcal C}u_{\rm FS}^2\,dy}\right.$$
$$\left.-\,\frac{4\,p\,\mu_{\rm FS}}{p+2}\,\frac{\int_{\mathcal C}u_{\rm FS}\,\psi_0\,dy}{\int_{\mathcal C}u_{\rm FS}^2\,dy}\right]\,,$$

because we notice that the terms involving k_ψ cancel. Hence, using (13) and (14), we have found that

$$\tau'(\mu_{\rm FS})=\frac{p-2}{p+2}+c_{p,d}\left[-\frac{4\,p^2\,\mu_{\rm FS}^2}{(p+2)^2}+\frac{2\,(p-1)\,p^3\,\mu_{\rm FS}^2}{(p+2)\,(3\,p-2)}\right.$$
$$\left.-\,\frac{2\,(p-2)}{p+2}\,\mu_{\rm FS}\,\frac{A_0}{\alpha}-\frac{4\,p\,\mu_{\rm FS}}{p+2}\,\frac{B_0}{\alpha}\,\frac{\mathsf b_{0,1}}{\mathsf I_2}\right]$$
$$=\frac{p-2}{p+2}-\,c_{p,d}\left[\frac{8\,p\,(d-1)}{(p-2)\,(p+2)^2}\right]^2\left[1+\frac{(p-1)\,p\,(p+2)}{2\,(p-2)}\,\frac{\mathsf b_{0,1}}{\mathsf I_2}\right].$$

Here the coefficient $\mathsf b_{0,1}$ is given by $\mathsf b_{0,1}:=\int_{\mathbb R}\chi_{0,2\,p-3}\,w\,ds$. Using (16), we obtain that

$$\tau'(\mu_{\rm FS})=\frac{p-2}{p+2}+\,\frac{16\,p^2\,(d-1)^2}{(p-2)\,(p+2)^3}\,c_{p,d}\,. \tag{22}$$

3.6. Expansion of $\nu(\mu)$ around $\mu_{\rm FS}$

Let us consider $\nu(\mu):=\|u_\mu\|^2_{\mathrm L^2(\mathcal C)}/\|u_\mu\|^2_{\mathrm L^p(\mathcal C)}$. Again we can write

$$\nu(\mu)-\nu(\mu_{\rm FS})=\left(\frac{\|u_{\mu,*}\|^2_{\mathrm L^2(\mathcal C)}}{\|u_{\mu,*}\|^2_{\mathrm L^p(\mathcal C)}}-\frac{\|u_{\rm FS}\|^2_{\mathrm L^2(\mathcal C)}}{\|u_{\rm FS}\|^2_{\mathrm L^p(\mathcal C)}}\right)+\left(\nu(\mu)-\frac{\|u_{\mu,*}\|^2_{\mathrm L^2(\mathcal C)}}{\|u_{\mu,*}\|^2_{\mathrm L^p(\mathcal C)}}\right)\,.$$

With $\beta=\frac{p-2}{2}\,\sqrt\mu$, using expressions that can be found in Appendix A.1, we see that

$$\nu_*(\mu):=\frac{\|u_{\mu,*}\|^2_{\mathrm L^2(\mathcal C)}}{\|u_{\mu,*}\|^2_{\mathrm L^p(\mathcal C)}}=\beta^{-\frac{p-2}{p}}\,\frac{\mathsf I_2}{\mathsf I_p^{2/p}}=\kappa_p\,\mu^{-\frac{p-2}{2\,p}}\,, \tag{23}$$

with $\kappa_p=\left(\frac{p+2}{4}\right)^{\frac2p}\left(\frac{2\,\sqrt\pi}{p-2}\,\frac{\Gamma\big(\frac{2}{p-2}\big)}{\Gamma\big(\frac{2}{p-2}+\frac12\big)}\right)^{\frac{p-2}{p}}$, and hence

$$\nu_*'(\mu_{\rm FS})=\lim_{\mu\to\mu_{\rm FS}}\frac{1}{\mu-\mu_{\rm FS}}\left(\frac{\|u_{\mu,*}\|^2_{\mathrm L^2(\mathcal C)}}{\|u_{\mu,*}\|^2_{\mathrm L^p(\mathcal C)}}-\frac{\|u_{\rm FS}\|^2_{\mathrm L^2(\mathcal C)}}{\|u_{\rm FS}\|^2_{\mathrm L^p(\mathcal C)}}\right)=-\frac{p-2}{2\,p}\,\frac{\nu_*(\mu_{\rm FS})}{\mu_{\rm FS}}\,.$$

If $u_\mu = u_{\mu,*} + \varepsilon\,\varphi + \varepsilon^2\,\psi + o(\varepsilon^2)$, where φ, ψ have been chosen in Section 3.2, after a Taylor expansion we find that

$$\nu(\mu) = \nu_*(\mu)\left[1 + \varepsilon^2\left(\frac{\int_{\mathcal C}\varphi^2\,dy}{\int_{\mathcal C}u_{\mu,*}^2\,dy} - (p-1)\,\frac{\int_{\mathcal C}u_{\mu,*}^{p-2}\,\varphi^2\,dy}{\int_{\mathcal C}u_{\mu,*}^p\,dy}\right)\right.$$
$$\left. + 2\,\varepsilon^2\,\frac{\int_{\mathcal C}u_{\mu,*}\,\psi\,dy}{\int_{\mathcal C}u_{\mu,*}^2\,dy} - 2\,\varepsilon^2\,\frac{\int_{\mathcal C}u_{\mu,*}^{p-1}\,\psi\,dy}{\int_{\mathcal C}u_{\mu,*}^p\,dy} + o(\varepsilon^2)\right]\,.$$

Again we may notice that the terms involving k_ψ cancel and, based on (13) and (14), we arrive at

$$\begin{aligned}\frac{\nu'(\mu_{\rm FS})}{\nu_*(\mu_{\rm FS})} &= -\,\frac{p-2}{2\,p\,\mu_{\rm FS}} + c_{p,d}\left[p\,\mu_{\rm FS}\left(\frac{2}{p+2} - \frac{p\,(p-1)}{3\,p-2}\right)\right.\\ &\qquad\left. + 2\left(\frac{B_0}{\alpha}\,\frac{\mathsf b_{0,1}}{\mathsf I_2} - \frac{B_0}{\alpha}\,\frac{\mathsf b_{0,p-1}}{\mathsf I_p}\right)\right]\\ &= -\,\frac{p-2}{2\,p\,\mu_{\rm FS}} + c_{p,d}\left[p\,\mu_{\rm FS}\left(\frac{2}{p+2} - \frac{p\,(p-1)}{3\,p-2}\right)\right.\\ &\qquad\left. + 2\,\frac{B_0}{\alpha}\left(\frac{\mathsf b_{0,1}}{\mathsf I_2} + \frac{p-2}{3\,p-2}\right)\right]\,.\end{aligned}$$

Claim 3 *At the bifurcation point $\mu = \mu_{\rm FS}$ we get the following inequalities.*

$$\frac{\nu_*'(\mu_{\rm FS})}{\nu_*(\mu_{\rm FS})} + \frac{\tau_*'(\mu_{\rm FS})}{\mu_{\rm FS} + \tau_*(\mu_{\rm FS})} = \frac{\nu'(\mu_{\rm FS})}{\nu_*(\mu_{\rm FS})} + \frac{\tau'(\mu_{\rm FS})}{\mu_{\rm FS} + \tau_*(\mu_{\rm FS})} = 0\,.$$

Proof. Recall that $\mathcal Q_\mu[u_{\mu,*}] = \nu_*(\mu)\,(\tau_*(\mu) + \mu)$ and $\mathcal Q_\mu[u_\mu] = \nu(\mu)\,(\tau(\mu) + \mu)$, so that

$$\frac{1}{Q_{\mu_{\rm FS}}[u_{\rm FS}]}\,\frac{d}{d\mu}\,Q_\mu[u_{\mu,*}]_{|\mu=\mu_{\rm FS}} = \frac{\nu_*'(\mu_{\rm FS})}{\nu_*(\mu_{\rm FS})} + \frac{1 + \tau_*'(\mu_{\rm FS})}{\mu_{\rm FS} + \tau_*(\mu_{\rm FS})} = 0$$

according to (21) and (23), and

$$\frac{1}{Q_{\mu_{\rm FS}}[u_{\rm FS}]}\,\frac{d}{d\mu}\,Q_\mu[u_\mu]_{|\mu=\mu_{\rm FS}} = \frac{\nu'(\mu_{\rm FS})}{\nu(\mu_{\rm FS})} + \frac{1 + \tau'(\mu_{\rm FS})}{\mu_{\rm FS} + \tau(\mu_{\rm FS})}\,.$$

According to (20), these two quantities are equal, thus proving the result. Alternatively, the identity can be proved directly using the expressions of τ' and ν' established in Sections 3.5 and 3.6. □

3.7. Reparametrization of the branch for $\theta < 1$

Now we are in position to study the local behavior of the branch of the solutions to (6) parametrized by μ close to the bifurcation point, that is, for μ in a neighborhood of $\mu_{\rm FS}$. More precisely, we are interested in the monotonicity of $\mu \mapsto \Lambda^\theta(\mu)$ and the behavior

of $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ in a neighborhood of $\mu = \mu_{\rm FS}$. According to (8), we know that $\Lambda^\theta(\mu) = \theta\,\mu - (1-\theta)\,\tau(\mu)$, so that

$$(\Lambda^\theta)' = \theta - (1-\theta)\,\tau'\,,$$

can be computed at $\mu = \mu_{\rm FS}$ using the expression of $\tau'(\mu_{\rm FS})$ that has been computed in Section 3.5. Hence we find that

$$(\Lambda^\theta)'(\mu_{\rm FS}) = \theta\,(1+\tau'(\mu_{\rm FS})) - \tau'(\mu_{\rm FS})\,.$$

Claim 4 *If $\tau'(\mu_{\rm FS})$ is positive, then we have that $\frac{d}{d\mu}\,\Lambda^\theta(\mu_{\rm FS}) < 0$ if and only*

$$\theta < \vartheta_2(p,d) := \frac{\tau'(\mu_{\rm FS})}{1+\tau'(\mu_{\rm FS})}\,.$$

Notice that with this definition, $\vartheta_2(p,d)$ is defined for any $p \in (2,2^*)$ and any $d \geq 2$. As long as $\vartheta(p,d) < \vartheta_2(p,d)$, $(\Lambda^\theta)'(\mu_{\rm FS})$ is negative if $\theta \in (\vartheta(p,d), \vartheta_2(p,d))$. In all numerical examples that are under consideration in this paper, we find that $\tau'(\mu_{\rm FS})$ is positive. This is of course automatically the case if $c_{p,d}$ itself is positive, because of (22).

We recall that $J^\theta(\mu) := \theta^\theta\,\left(\mu + \tau(\mu)\right)^\theta\,\nu(\mu)$ and $\Lambda^\theta(\mu) = \theta\,\mu - (1-\theta)\,\tau(\mu)$. Hence

$$(\log J^\theta)' = \frac{\nu'}{\nu} + \theta\,\frac{1+\tau'}{\mu+\tau}$$

and a similar formula holds for J^θ_*. At $\mu = \mu_{\rm FS}$, we can use Claim 3 and get

$$(\log J^\theta)'(\mu_{\rm FS}) = \frac{\theta\,(1+\tau'(\mu_{\rm FS})) - \tau'(\mu_{\rm FS})}{\mu_{\rm FS} + \tau(\mu_{\rm FS})} = \frac{(\Lambda^\theta)'(\mu_{\rm FS})}{\mu_{\rm FS} + \tau(\mu_{\rm FS})}\,.$$

Let us define

$$\delta^\theta := \frac{(J^\theta)'}{(\Lambda^\theta)'} - \frac{(J^\theta_*)'}{(\Lambda^\theta_*)'}\,.$$

Claim 5 *Assuming that remainder terms are all of lower order in the above computations, we have*

$$\delta^\theta(\mu_{\rm FS}) = 0\,.$$

In other words, the parametric curves $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ and $\mu \mapsto (\Lambda^\theta_*(\mu), J^\theta_*(\mu))$ are tangent at $\mu = \mu_{\rm FS}$. It remains to decide the relative position of the two branches in a neighborhood of the bifurcation point. In order to do so, let us define the function

$$\xi^\theta := \frac{((\Lambda^\theta)')^2}{J^\theta}\,\left[\frac{1}{(\Lambda^\theta)'}\,\frac{d}{d\mu}\left(\frac{(J^\theta)'}{(\Lambda^\theta)'}\right) - \frac{1}{(\Lambda^\theta_*)'}\,\frac{d}{d\mu}\left(\frac{(J^\theta_*)'}{(\Lambda^\theta_*)'}\right)\right]\,.$$

which, up to a positive factor, is the difference of the second parametric derivatives of J^θ and J^θ_*.

Claim 6 *Assume that $\vartheta_2(p,d)\geq\vartheta(p,d)$. The parametric curve $\mu\mapsto(\Lambda^\theta(\mu),J^\theta(\mu))$ is concave and below the parametric curve $\mu\mapsto(\Lambda^\theta_*(\mu),J^\theta_*(\mu))$ for $\mu-\mu_{\rm FS}>0$, small, if $\theta>\vartheta_2(p,d)$ and $\xi^\theta(\mu_{\rm FS})<0$. The parametric curve $\mu\mapsto(\Lambda^\theta(\mu),J^\theta(\mu))$ is above the parametric curve $\mu\mapsto(\Lambda^\theta_*(\mu),J^\theta_*(\mu))$ for $\mu-\mu_{\rm FS}>0$, small, if $\theta<\vartheta_2(p,d)$ and $\xi^\theta(\mu_{\rm FS})<0$.*

Now let us discuss the range of positivity of ξ^θ. An elementary computation shows that

$$\xi^\theta=\frac{(J^\theta)''}{J^\theta}-\frac{(J^\theta)'}{(\Lambda^\theta)'}\,\frac{(\Lambda^\theta)''}{J^\theta}-\frac{(J^\theta_*)''}{J^\theta_*}\left(\frac{(\Lambda^\theta)'}{(\Lambda^\theta_*)'}\right)^2\,,$$

because $(\Lambda^\theta_*)''=0$. By definition of $\Lambda^\theta(\mu)=\theta\,\mu-(1-\theta)\,\tau(\mu)$ and $\vartheta_2(p,d)$, we get that $(\Lambda^\theta)''(\mu_{\rm FS})=-\,(1-\theta)\,\tau''(\mu_{\rm FS})$ and $(\Lambda^\theta)'(\mu_{\rm FS})=\frac{\theta-\vartheta_2(p,d)}{1-\vartheta_2(p,d)}$. From (21) and $\Lambda^\theta_*(\mu)=\theta\,\mu-(1-\theta)\,\tau_*(\mu)$, we get that $(\Lambda^\theta_*)'=\frac{2\,p\,\theta-(p-2)}{p+2}$. From (21) and (23), we know also that $J^\theta_*(\mu)=\kappa_p\,\left(2\,p\,\theta/(p+2)\right)^\theta\,\mu^{\theta-\frac{p-2}{2\,p}}$, so that $\frac{(J^\theta_*)'}{J^\theta_*}=\frac{2\,p\,\theta-(p-2)}{2\,p\,\mu}$ and $(\log J^\theta_*)''=-\frac{2\,p\,\theta-(p-2)}{2\,p\,\mu^2}$. As a consequence of Claim 5, we can write that

$$\frac{(J^\theta)'(\mu_{\rm FS})}{J^\theta(\mu_{\rm FS})}=\frac{(J^\theta_*)'(\mu_{\rm FS})}{J^\theta_*(\mu_{\rm FS})}\,\frac{(\Lambda^\theta)'(\mu_{\rm FS})}{(\Lambda^\theta_*)'(\mu_{\rm FS})}\,.$$

According to (20), we have the identity

$$-\frac14\,(p^2-4)\,c_{p,d}=\left(\log\frac{J^\theta}{J^\theta_*}\right)''(\mu_{\rm FS})+(1-\theta)\left(\log\frac{\mu+\tau(\mu)}{\mu+\tau_*(\mu)}\right)''_{|\mu=\mu_{\rm FS}}\,,$$

which allows us to compute

$$\begin{aligned}
&\frac{(J^\theta)''(\mu_{\rm FS})}{J^\theta(\mu_{\rm FS})}\\
&=\;-\,\frac14\,(p^2-4)\,c_{p,d}+\left(\frac{(J^\theta)'(\mu_{\rm FS})}{J^\theta(\mu_{\rm FS})}\right)^2+(\log J^\theta_*)''(\mu_{\rm FS})\\
&\quad-(1-\theta)\left[\frac{p+2}{2\,p\,\mu_{\rm FS}}\,\tau''(\mu_{\rm FS})-\left(\frac{p+2}{2\,p\,\mu_{\rm FS}}\,(1+\tau'(\mu_{\rm FS}))\right)^2+\frac{1}{\mu_{\rm FS}^2}\right]\\
&=\;-\,\frac14\,(p^2-4)\,c_{p,d}+\left(\frac{(J^\theta_*)'(\mu_{\rm FS})}{J^\theta_*(\mu_{\rm FS})}\right)^2\left(\frac{(\Lambda^\theta)'(\mu_{\rm FS})}{(\Lambda^\theta_*)'(\mu_{\rm FS})}\right)^2+(\log J^\theta_*)''(\mu_{\rm FS})\\
&\quad-(1-\theta)\left[\frac{p+2}{2\,p\,\mu_{\rm FS}}\,\tau''(\mu_{\rm FS})-\left(\frac{p+2}{2\,p\,\mu_{\rm FS}}\,(1+\tau'(\mu_{\rm FS}))\right)^2+\frac{1}{\mu_{\rm FS}^2}\right].
\end{aligned}$$

Because of Claim 5, we also have

$$-\frac{(J^\theta)'(\mu_{\rm FS})}{(\Lambda^\theta)'(\mu_{\rm FS})}\,\frac{(\Lambda^\theta)''(\mu_{\rm FS})}{J^\theta(\mu_{\rm FS})}=-\,\frac{(J^\theta_*)'(\mu_{\rm FS})}{J^\theta_*(\mu_{\rm FS})}\,\frac{(\Lambda^\theta)''(\mu_{\rm FS})}{(\Lambda^\theta_*)'(\mu_{\rm FS})}$$

$$= \frac{p+2}{2\,p\,\mu_{\rm FS}}\,(1-\theta)\,\tau''(\mu_{\rm FS})\,.$$

Collecting the above identities, we can compute the value of $\xi^\theta(\mu_{\rm FS})$ as

$$\begin{aligned}\xi^\theta(\mu_{\rm FS}) = & -\frac{1}{4}\,(p^2-4)\,c_{p,d}\\ & + (\log J_*^\theta)''(\mu_{\rm FS})\left[1-\left(\frac{(\Lambda^\theta)'(\mu_{\rm FS})}{(\Lambda_*^\theta)'(\mu_{\rm FS})}\right)^2\right]\\ & - \frac{1-\theta}{\mu_{\rm FS}^2}\left[1-\left(\frac{p+2}{2\,p}\,(1+\tau'(\mu_{\rm FS}))\right)^2\right].\end{aligned}$$

The cancellation of the terms involving $\tau''(\mu_{\rm FS})$ is a remarkable fact. By definition of $\vartheta_2(p,d)$, we get

$$\begin{aligned}\xi^\theta(\mu_{\rm FS}) = & -\frac{1}{4}\,(p^2-4)\,c_{p,d}\\ & - \frac{2\,p\,\theta-(p-2)}{2\,p\,\mu_{\rm FS}^2}\left[1-\left(\frac{p+2}{2\,p\,\theta-(p-2)}\,\frac{\theta-\vartheta_2(p,d)}{1-\vartheta_2(p,d)}\right)^2\right]\\ & - \frac{1-\theta}{\mu_{\rm FS}^2}\left[1-\left(\frac{p+2}{2\,p}\,\frac{1}{1-\vartheta_2(p,d)}\right)^2\right]\end{aligned}$$

and finally arrive at

$$\xi^\theta(\mu_{\rm FS}) = -\frac{1}{4}\,(p^2-4)\,c_{p,d} + \frac{p+2}{4\,p^2\,\mu_{\rm FS}^2}\,\frac{(1-\theta)\,(2\,p\,\vartheta_2(p,d)-(p-2))^2}{(1-\vartheta_2(p,d))^2\,(2\,p\,\theta-(p-2))}\,.$$

At this point, we can observe that $\vartheta(p,d) \ge \frac{p-2}{2\,p}$. The reader is then invited to check that the function $\theta \mapsto \xi^\theta(\mu_{\rm FS})$ is nonincreasing on $[\vartheta(p,d),1]$ and

$$\xi^{\vartheta_2(p,d)}(\mu_{\rm FS}) = -\frac{1}{4}\,(p^2-4)\,c_{p,d} + \frac{p+2}{4\,p^2\,\mu_{\rm FS}^2}\,\frac{2\,p\,\vartheta_2(p,d)-(p-2)}{1-\vartheta_2(p,d)} = 0$$

because of (22). Recall that the positivity of $c_{p,d}$ is required in (19).

Claim 7 *If $\vartheta_2(p,d) \le \vartheta(p,d)$, $\xi^\theta(\mu_{\rm FS})$ is negative for any $\theta \in (\vartheta(p,d),1]$. Otherwise, if $c_{p,d}$ is positive, $\xi^\theta(\mu_{\rm FS})$ is positive if $\theta \in [\vartheta(p,d),\vartheta_2(p,d))$ and negative if $\theta \in (\vartheta_2(p,d),1]$.*

Summarizing, the expansion (11) and the subsequent computations are valid, and make sense, as soon as the coefficient $c_{p,d}$, whose expression is established in Section 3.3, is positive. Then for any $\theta \in (\max\{\vartheta(p,d),\vartheta_2(p,d)\},1)$, the curve of the energies of the non-symmetric solutions of the Euler-Lagrange equations is concave, nondecreasing as a function of Λ in a neighborhood of the bifurcation point, and below the energies of the symmetric functions. If $\vartheta_2(p,d) > \vartheta(p,d)$, then the curve of the energies of the non-symmetric solutions is above the energies of symmetric functions in a neighborhood of the bifurcation point if $\theta \in [\vartheta(p,d),\vartheta_2(p,d))$. To decide whether $c_{p,d}$ is positive or not relies on numerical computations. However, the estimate of Section 3.4 shows that this occurs at least in a large subinterval of $(2,2^*)$.

4. Numerical results and the two *scenarii*

4.1. Symmetric and non-symmetric branches, and their asymptotic behavior

In [12] the branches of solutions which bifurcate from the branches of symmetric solutions at their smallest possible bifurcation point have been computed numerically. For completeness, we start by exposing some of these numerical results, which are the main motivation of the present paper. The branch of symmetric solutions is explicit. The branch $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ bifurcates from the symmetric ones at $\mu = \mu_{\rm FS}$ and is computed numerically. The algorithm is based on descent techniques and on an iteration scheme which allows us to compute the branches of solutions by continuation. We carried out the computations for dimension $d = 5$ and various values of p and θ. We have of course no guarantee that the solutions that we have computed are the optimal ones, but at least the values that we have found are fully compatible with what is expected. In particular, the curve of the computed estimates of the best constant is a concave increasing function of Λ, which can reasonably be expected to coincide with $\Lambda \mapsto \mathsf{K}_{\rm CKN}(\theta, \Lambda, p)$. Moreover, when θ approaches $\vartheta(p, d)$ from above, the curve $\Lambda \mapsto 1/\mathsf{K}_{\rm CKN}(\theta, \Lambda, p)$ approaches $\Lambda \mapsto \max\{\mathsf{K}^*_{\rm CKN}(\theta, \Lambda, p), \mathsf{K}_{\rm GN}\}$. Last but not least, the asymptotics predicted in Proposition 1 are not only correct (dotted lines in Figs 1–3) but provide a good upper estimate of the curve in the whole range $\Lambda > 0$.

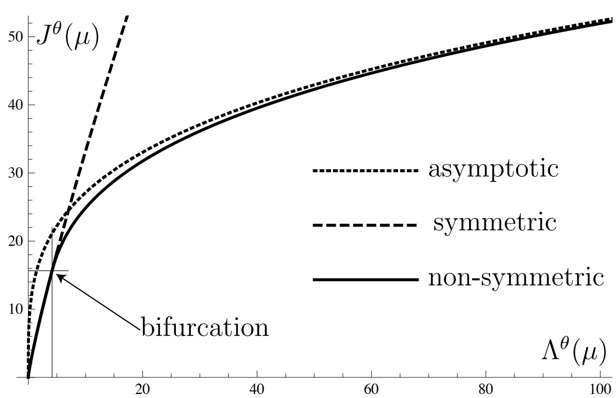

Figure 1. Parametric plot of $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ for $p = 2.8$, $d = 5$, $\theta = 1$. Non-symmetric solutions bifurcate from symmetric ones at a bifurcation point $\mu = \mu_{\rm FS}$ computed by V. Felli and M. Schneider. The branch behaves for large values of Λ as predicted by F. Catrina and Z.-Q. Wang.

4.2. Two scenarii

The branch of symmetric minimal solutions, (Λ, u^*_Λ), is known explicitly and is monotone in energy, that is, the function $\Lambda \mapsto \mathcal{Q}^\theta_\Lambda(u^*_\Lambda)$ is monotone increasing in $(0, +\infty)$. In the computations described in [12] we observe that the branch of non-symmetric solutions,

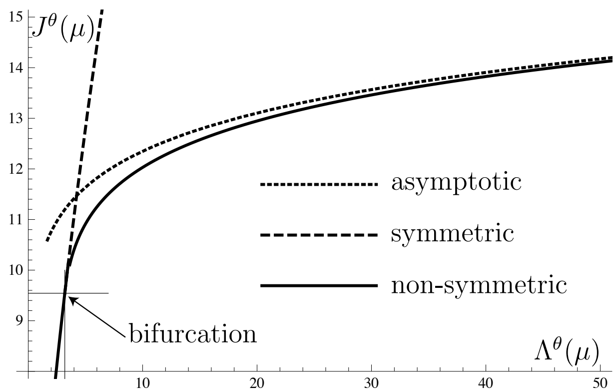

Figure 2. Parametric plot of $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ for $p = 2.8$, $d = 5$, $\theta = 0.8$. The behavior is similar to the case $\theta = 1$ up to the reparametrization described in Section 2.3, while the asymptotic behavior of the branch for large values of Λ is in agreement with the results of Section 2.4.

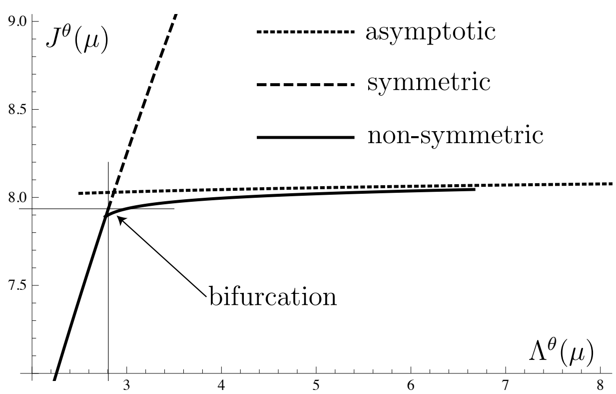

Figure 3. Parametric plot of $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ for $p = 2.8$, $d = 5$, $\theta = 0.72 > \vartheta(2.8, 5) \approx 0.7143$. The asymptotic behavior of the branch for large values of Λ is in agreement with the results of Section 2.4. We may however notice that the bifurcation seems not to occur where expected. More detailed computation in the Section 4.3 will shed light on this phenomenon.

$(\Lambda^\theta(\mu), \mathcal{Q}^\theta_\mu[u_\mu])$, is monotone for some values of θ (for instance for $\theta = 1$), but not always. More concretely, for certain values of p and d, the numerical results show that there exists an exponent $\vartheta_1 = \vartheta_1(p,d) \in (\vartheta(p,d), 1)$ such that for any $\theta \in [\vartheta_1, 1]$ the branch $(\Lambda^\theta(\mu), \mathcal{Q}^\theta_\mu[u_\mu])$ is monotone increasing. But when $\vartheta(p,d) < \theta < \vartheta_1(p,d)$, a dramatic change occurs: see Figs. 5 and 6. For the values of p and d that have been considered numerically in those figures, the branch is not monotone anymore for $\mu > \mu_{\rm FS}$, thus producing non-symmetric solutions and candidates for optimal functions in the inequalities for values of $\Lambda < \Lambda^\theta(\mu_{\rm FS})$. This phenomenon provides an explanation for the results proved in [3] using rigorous *a priori* estimates.

The limiting case $\theta = \vartheta(p,d)$ is very interesting: see Figs. 6 and 9. Let us define

$\Lambda^*_{\rm GN}(p,d)$ and $\mu_{\rm GN}$ such that

$$\mathsf{K}_{\rm GN} = \mathsf{K}^*_{\rm CKN}(\vartheta(p,d), \Lambda^*_{\rm GN}(p,d), p) \quad \text{and} \quad \Lambda^{\vartheta(p,d)}_*(\mu_{\rm GN}) = \Lambda^*_{\rm GN}(p,d)\,.$$

Whether $\mathsf{K}_{\rm GN}$ is larger or smaller than $\mathsf{K}^*_{\rm CKN}(\vartheta(p,d), \Lambda_{\rm FS}(p,\vartheta(p,d)), p)$ determines, at least in the framework of our computations, whether θ is smaller than $\vartheta_1(p,d)$ or not. This has been observed in [13] and theoretical consequences have been established in [3], in the limit regime $p \to 2$. Before going further, let us observe that $\vartheta_1(p,d)$ is an exponent associated to a global property of the branch.

Based on our numerical computations, we are now in position to formulate the following alternative.

Scenario 1. If $\mathsf{K}_{\rm GN} \le \mathsf{K}^*_{\rm CKN}(\vartheta(p,d), \Lambda_{\rm FS}(p,\vartheta(p,d)), p)$, then for all $\theta \ge \vartheta(p,d)$, the optimal functions are symmetric for any $\Lambda \in (0, \Lambda_{\rm FS}(p,\theta)]$ and the branch of non-symmetric solutions is optimal for any $\Lambda > \Lambda_{\rm FS}(p,\theta)$. Such solutions exist for arbitrarily large values of Λ if $\theta > \vartheta(p,d)$: see Figs. 7 and 8, but may exist only for a finite range of Λ if $\theta = \vartheta(p,d)$ and $\mathsf{K}_{\rm GN} < \mathsf{K}^*_{\rm CKN}(\vartheta(p,d), \Lambda_{\rm FS}(p,\vartheta(p,d)), p)$: see Fig. 9.

Scenario 2. If $\mathsf{K}_{\rm GN} > \mathsf{K}^*_{\rm CKN}(\vartheta(p,d), \Lambda_{\rm FS}(p,\vartheta(p,d)), p)$, there exists $\vartheta_1 = \vartheta_1(p,d) \in (\vartheta(p,d), 1)$ such that for any $\theta \in [\vartheta_1, 1]$ the branch is monotone increasing. We further observe numerically that $\vartheta_1 = \vartheta_2$ (see Fig. 12), where ϑ_2 has been defined in Section 3.7: for any $\theta \in [\vartheta(p,d), \vartheta_2(p,d))$, we know that the branch of non-symmetric functions is decreasing in a neighborhood of the bifurcation point, but also has a larger energy than the symmetric solutions of the Euler-Lagrange equation (for the same value of Λ). Hence, in the critical case $\theta = \vartheta(p,d)$ we have $\Lambda^*_{\rm GN}(p,d) < \Lambda_{\rm FS}(p,\vartheta(p,d))$ and for any Λ in the range $(0, \Lambda^*_{\rm GN}] := \{\Lambda^{\vartheta(p,d)}_*(\mu) \;:\; \mu \in (0, \mu_{\rm GN}]\}$, the optimal functions are symmetric and $\mathsf{K}_{\rm CKN}(\vartheta(p,d), \Lambda, p) = \mathsf{K}^*_{\rm CKN}(\vartheta(p,d), \Lambda, p)$. See Fig. 6. Moreover, in the language of the concentration-compactness method, according to [16], for any $\Lambda > \Lambda^*_{\rm GN}(p,d)$ the optimal constant is determined by the *problem at infinity* and $\mathsf{K}_{\rm CKN}(\vartheta(p,d), \Lambda, p) = \mathsf{K}_{\rm GN}$. From the viewpoint of the reparametrized branch, we numerically observe that both $\mu \mapsto \Lambda^{\vartheta(p,d)}(\mu)$ and $\mu \mapsto J^{\vartheta(p,d)}(\mu)$ are decreasing for $\mu > \mu_{\rm FS}$, at least for the values of p for which computations have been done. In the subcritical case corresponding to $\vartheta(p,d) < \theta < \vartheta_2(p,d)$, the reparametrized branch $(\Lambda^\theta, J^\theta)$ is not monotone. Numerically we observe that it is monotone for $\theta > \vartheta_2(p,d)$, hence suggesting that $\vartheta_1(p,d) = \vartheta_2(p,d)$ (see Fig. 12).

Altogether, based on our numerical observations, what decides between *Scenario 1* and *Scenario 2* is the relative value of $\mathsf{K}_{\rm GN}$ and $\mathsf{K}^*_{\rm CKN}(\vartheta(p,d), \Lambda_{\rm FS}(p,\vartheta(p,d)), p)$. Equality of these two optimal constants determines a value of $p = p^\star(d)$. Numerically we find that $p^\star(5) \approx 3.001$ and $\vartheta(p^\star(5), 5) = \vartheta_1(p^\star(5), 5) \approx 0.834$. For $p \in [p^\star(d), 2^*)$, only *Scenario 1* occurs (numerical observation). For $p \in (2, p^\star(d))$ we have $\vartheta(p,d) > \vartheta_1(p,d)$ and *Scenario 2* occurs. More precisely, the fact that the branch cannot be globally monotone increasing if $\theta < \vartheta_2(p,d)$ is a consequence of Section 3.7 while the fact that

the branch is monotone increasing if $\theta > \vartheta_2(p,d)$ is a numerical observation. These results are fully consistent with the ones of [3] and [13]. Now let us give some details.

4.3. Bifurcations and qualitative dependence in θ

In Fig. 3, a careful inspection shows that the symmetric and the non-symmetric branches of solutions differ for values of Λ strictly less than $\Lambda^\theta(\mu_{\rm FS})$. This is not the case for θ close enough to 1: see Fig. 4, but very clear on Fig. 5. When θ approaches $\vartheta(p,d)$, the

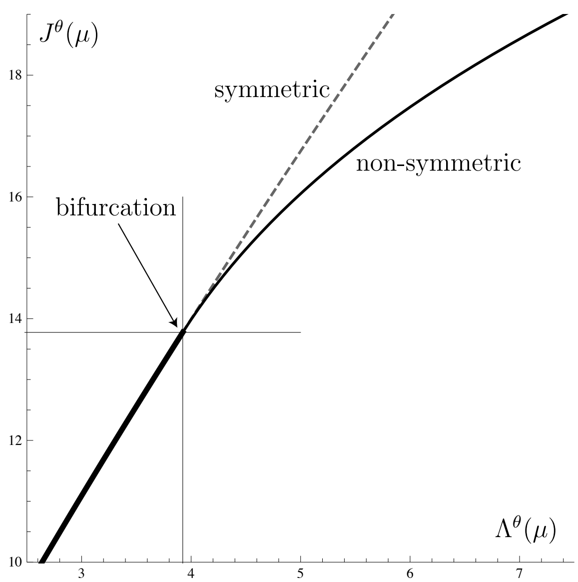

Figure 4. Parametric plot of $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ for $p = 2.8$, $d = 5$, $\theta = 0.95$ close to the bifurcation point. For $\Lambda \le \Lambda^\theta(\mu_{\rm FS})$, all solutions are symmetric, while for $\Lambda > \Lambda^\theta(\mu_{\rm FS})$, non-symmetric solutions provide a better constant in the interpolation inequalities, exactly as for the case $\theta = 1$.

branch (locally) converges to its limit: see Fig. 6. Figs. 5 and 6 correspond to *Scenario 2* (for $\vartheta \le \theta < \vartheta_1$), that is, to the case

$$\mathsf{K}_{\rm GN} > \mathsf{K}^*_{\rm CKN}(\vartheta(p,d), \Lambda_{\rm FS}(p, \vartheta(p,d)), p)\,.$$

In other words, this means that $p < p^\star(d)$.

The case $p > p^\star(d)$, *i.e., Scenario 1,* also occurs and corresponding plots are shown in Figs. 7–9. There we take $d = 5$, $p = 3.15 \ge p^\star(5) \approx 3.001$.

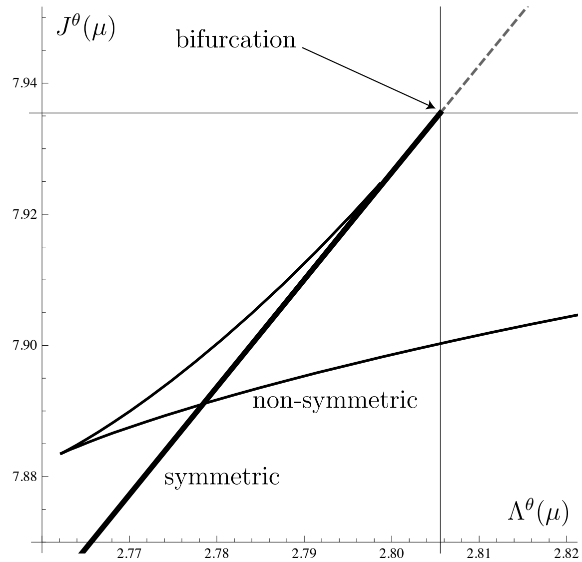

Figure 5. Parametric plot of $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ for $p = 2.8$, $d = 5$, $\theta = 0.72$ close to the bifurcation point. Non-symmetric solutions exist for $\Lambda < \Lambda^\theta(\mu_{\mathrm{FS}})$. There exists a value $\Lambda^*_{\mathrm{GN}} < \Lambda^\theta(\mu_{\mathrm{FS}})$ such that optimal functions are symmetric for any $\Lambda \in (0, \Lambda^*_{\mathrm{GN}})$ and are non-symmetric for $\Lambda > \Lambda^*_{\mathrm{GN}}$. When $\Lambda = \Lambda^*_{\mathrm{GN}}$, symmetric and non-symmetric optimal functions co-exist.

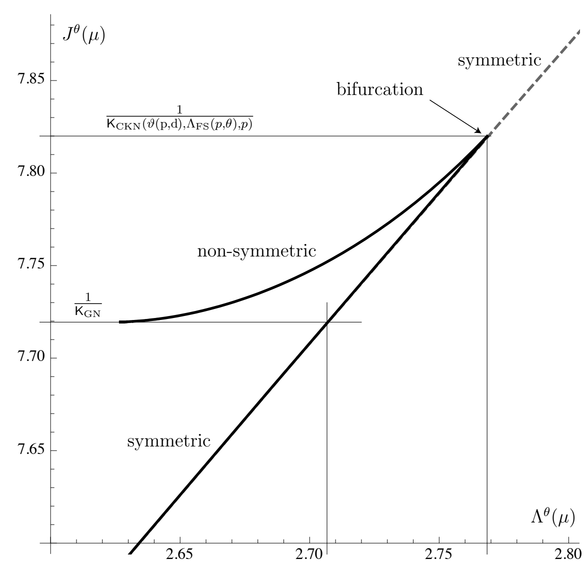

Figure 6. Critical case $\theta = \vartheta(p, d)$. Parametric plot of $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ for $p = 2.8$, $d = 5$, $\theta = \vartheta(2.8, 5) \approx 0.7143$ close to the bifurcation point. Non-symmetric solutions exist for $\Lambda < \Lambda^\theta(\mu_{\mathrm{FS}})$ but, at least for the chosen values of p and d, are never optimal functions for the interpolation inequalities. There exists a value $\Lambda^*_{\mathrm{GN}} < \Lambda^\theta(\mu_{\mathrm{FS}})$ such that optimal functions exists and are symmetric for any $\Lambda \in (0, \Lambda^*_{\mathrm{GN}}]$ and do not exist for $\Lambda > \Lambda^*_{\mathrm{GN}}$. Moreover, $\Lambda^*_{\mathrm{GN}} = \Lambda^\theta(\mu_{\mathrm{GN}})$ with the notations of Section 2.4 and $\mathsf{K}_{\mathrm{CKN}}(\vartheta(p, d), \Lambda, p) = \mathsf{K}_{\mathrm{GN}}$ for any $\Lambda > \Lambda^*_{\mathrm{GN}}$.

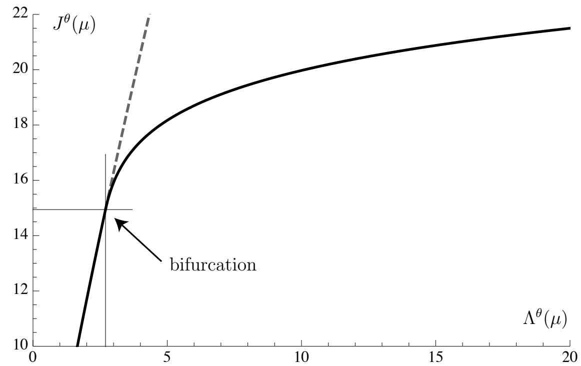

Figure 7. Parametric plot of $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ for $p = 3.15$, $d = 5$, $\theta = 1$. The behavior of the non-symmetric branch is similar to the one found in Fig. 1.

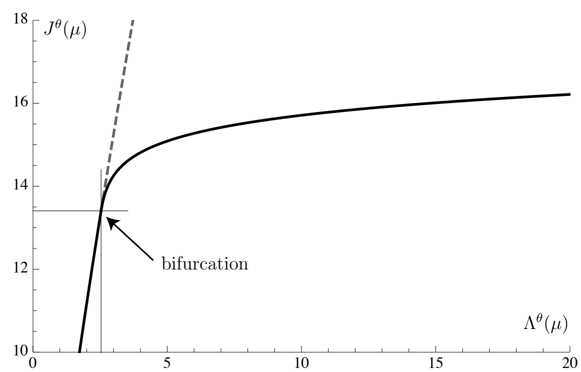

Figure 8. Parametric plot of $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ for $p = 3.15$, $d = 5$, $\theta = 0.95$. The behavior of the non-symmetric branch is still similar to the one found in Fig. 4.

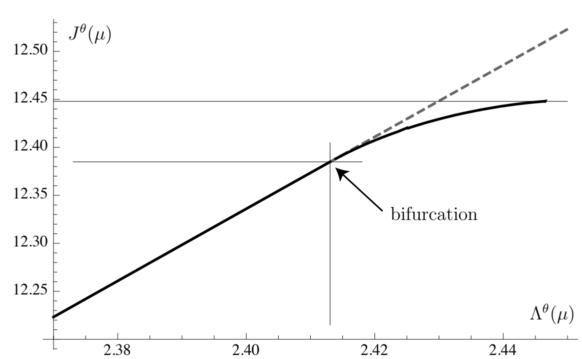

Figure 9. Parametric plot of $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ for $p = 3.15$, $d = 5$, $\theta = \vartheta(3.15, 5) \approx 0.9127$, that is for the critical value of θ.

In Section 3 we proved that the symmetric and the non-symmetric branches of solutions are always tangent at $\Lambda_{\rm FS}$. What happens in a neighborhood of the bifurcation point is therefore difficult to decide in view of the plots of the branches, especially when θ is in a neighborhood of $\vartheta_1(p,d)$. To illustrate this difficulty, we may for instance observe that figure Fig. 5 is an enlargement of Fig. 3. Hence, in *Scenario 1,* we have to discard the possibility of other *scenarii* than the ones described in Section 4.2 at least in a neighborhood of the bifurcation point.

The computations of Section 3 are formal for two reasons. We assume that the solutions in the non-symmetric branch correspond to the *ansatz* given by (11) and (17) up to various higher order terms, and we also require that $c_{p,d}$ is positive. This last condition has been discussed in Section 3.4, but can be checked numerically within the *ansatz*: we know that $c_{p,d}$ is positive and finite as long as $c_{p,d}^{\rm approx}$ is positive (see Fig. 10, and Fig. 11 for a discussion of the sign of $c_{p,d}^{\rm approx}$), and numerically we find that $c_{p,d}$ is always positive.

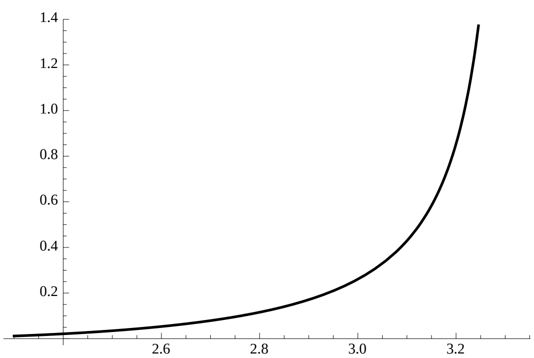

Figure 10. Computation of $c_{p,d}$ with $d=5$ as a function of p. We observe that the numerical solution is positive for any $p\in(2,10/3)$ where $10/3$ is the critical exponent (corresponding to the plain vertical line). The estimate of Section 3.4 corresponds to the dotted line and holds for $p\lesssim 3.2323$ (corresponding to the dotted vertical line).

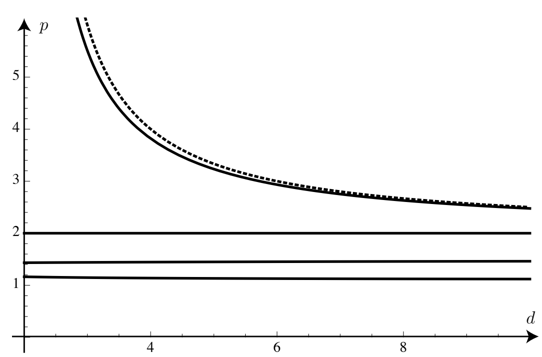

Figure 11. The plain curves are the zeros of $c_{p,d}^{\rm approx}$ as d varies in the interval $(2,10)$, except $p=2$ which is a singularity. The dotted curve is given by $d\mapsto 2\,d/(d-2)=2^*$. For a given d, $c_{p,d}^{\rm approx}$ is therefore positive for a large subinterval in p of $(2,2^*)$.

Under the above precautions, we know from Section 3 that there exists a number $\vartheta_2(p,d)$ such that the behavior of the branch in a neighborhood of the bifurcation point $\mu = \mu_{\rm FS}$ discriminates between two regimes corresponding to $\theta > \vartheta_2(p,d)$ and $\theta < \vartheta_2(p,d)$. When $\theta < \vartheta_2(p,d)$, we have $(\Lambda^\theta)'(\mu_{\rm FS} < 0$ and the contrary happens when $\theta > \vartheta_2(p,d)$. So, locally, the reparametrized branch is monotone increasing in a right neighborhood of $\mu_{\rm FS}$ if and only if $\theta > \vartheta_2(p,d)$. Since global monotonicity implies local monotonicity near the bifurcation point, if the numerical computations of the branches are consistent with the formal study of the bifurcation carried out in Section 3, then we should have that $\vartheta_1(p,d) = \vartheta_2(p,d)$. It is not easy to establish a qualitative property such as the monotonicity, but at least we observe in Fig. 12 that for $\theta = \vartheta(p,d)$ the range in p for which $\vartheta_2(p,d) \geq \vartheta(p,d)$ corresponds to the range in p for which the Gagliardo-Nirenberg constant compares well with the energy at the bifurcation point.

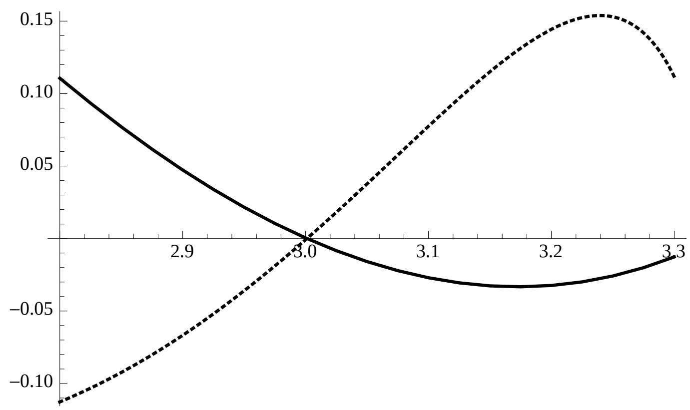

Figure 12. Comparison of the local and asymptotic criteria in the critical case $\theta = \vartheta(p,d)$ when $d = 5$. The dotted curve corresponds to the function $p \mapsto 1/\mathsf{K}_{\rm GN}(p,d) - J^{\vartheta(p,d)}(\mu_{\rm FS})$, that is, the difference of the asymptotic energy of the branch and the energy at the bifurcation point: when it is negative, this means that $\vartheta_1(p,d)$ is defined and larger than $\vartheta(p,d)$, so that *Scenario 2* takes place. When it is positive, this means that *Scenario 1* can be expected. The exponent $\vartheta_2(p,d)$ can be defined for any $p \in (2, 2^*)$. The plain curve represents $p \mapsto 5\,(\vartheta_2(p,d) - \vartheta(p,d))$ and positivity indicates that, at least locally around the bifurcation point, *Scenario 2* takes place. Hence the local (around the bifurcation point) and asymptotic (as $\Lambda \to +\infty$) criteria coincide.

5. Concluding remarks

In this paper we have established the asymptotic behavior of the branches for all $\theta \in (\vartheta(p,d), 1]$. There is a good agreement between this behavior and that of the numerical branches, which reinforces the conjecture that the computed branches contain the extremals for the Caffarelli-Kohn-Nirenberg inequalities.

We have also studied the precise behavior of the branches of non-symmetric

solutions near the first possible bifurcation point on the symmetric branch within the framework of a particular *ansatz*. By doing so, we have obtained the existence of a critical exponent $\vartheta_2(p,d)$ above which the branch is monotone, increasing and potentially optimal, and below which it is certainly not optimal in a neighborhood of the bifurcation point.

The final result of this paper is the comparison of the above criterion based on the local behavior of the branch near the bifurcation point and the criterion based on the asymptotic energy of the branch in the critical case $\theta = \vartheta(p,d)$, using numerical methods. They coincide, which gives solid grounds to the alternative that has been numerically observed:

Scenario 1. The non-symmetric branch is monotone increasing for any $\theta \in [\vartheta(p,d),1]$.

Scenario 2. The non-symmetric branch is monotone increasing for any $\theta \in (\vartheta_2(p,d),1]$ but it is not optimal near the bifurcation point if $\theta \in [\vartheta(p,d),\vartheta_2(p,d))$.

This suggests that no other *scenario* can take place, consistently with our numerical computations. Hence we arrive at the conclusion that *Scenario 1* takes place when $\vartheta_2(p,d) < \vartheta(p,d)$ and *Scenario 2* holds if $\vartheta_2(p,d) > \vartheta(p,d)$. The branches of solutions we have computed are likely to be optimal for the Caffarelli-Kohn-Nirenberg inequalities.

Appendix A. Some useful quantities

Appendix A.1. Computing integrals

We recall that $f(q) := \int_{\mathbb{R}} \frac{ds}{(\cosh s)^q}$ can be explicitly computed: $f(q) = \frac{\sqrt{\pi}\,\Gamma(\frac{q}{2})}{\Gamma(\frac{q+1}{2})}$. An integration by parts shows that $f(q+2) = \frac{q}{q+1}\,f(q)$. The following formulae are reproduced with no change from [6] (also see [11]). As in [6], with $w(s) = (\cosh s)^{-\frac{2}{p-2}}$, we can define

$$\mathsf{I}_q := \int_{\mathbb{R}} |w(s)|^q\,ds \quad \text{and} \quad \mathsf{J}_2 := \int_{\mathbb{R}} |w'(s)|^2\,ds\,.$$

Using the function f, we can compute $\mathsf{I}_2 = f(\frac{4}{p-2})$, $\mathsf{I}_p = f(\frac{2\,p}{p-2}) = f(\frac{4}{p-2}+2)$ and get the relations

$$\mathsf{I}_2 = \frac{\sqrt{\pi}\;\Gamma(\frac{2}{p-2})}{\Gamma(\frac{p+2}{2\,(p-2)})}\,, \quad \mathsf{I}_p = \frac{4\,\mathsf{I}_2}{p+2}\,, \quad \mathsf{J}_2 = \frac{4\,\mathsf{I}_2}{(p+2)\,(p-2)}\,.$$

As a special case, we have

$$\mathsf{I}_p := \int_{\mathbb{R}} (\cosh s)^{-\frac{2\,p}{p-2}}\;ds = \frac{4\,\sqrt{\pi}\,\Gamma\left(\frac{2}{p-2}\right)}{(p+2)\,\Gamma\left(\frac{p+2}{2\,(p-2)}\right)}\,.$$

Appendix A.2. Symmetric extremals and linearization

Consider $w(s)=(\cosh s)^{-\frac{2}{p-2}}$, which is the unique positive solution of

$$-(p-2)^2\,w''+4\,w-2\,p\,w^{p-1}=0$$

on $\mathbb{R}$, up to translations. The function $u(s):=\alpha\,w(\beta\,s)$ solves

$$-\,u''+\frac{4\,\beta^2}{(p-2)^2}\,u-\frac{2\,p\,\beta^2}{(p-2)^2}\,\alpha^{2-p}\,u^{p-1}=0\,. \tag{A.1}$$

With $\beta=\frac{p-2}{2}\sqrt{\mu}$ and $\alpha=(\frac{p}{2}\,\mu)^{\frac{1}{p-2}}$, $u=u_{\mu,*}$ is given by

$$u_{\mu,*}(s)=\left(\frac{p}{2}\,\mu\right)^{\frac{1}{p-2}}\left[\cosh\left(\frac{p-2}{2}\,\sqrt{\mu}\,s\right)\right]^{-\frac{2}{p-2}}\qquad\forall\,s\in\mathbb{R}$$

and solves (4).

Next we are interested in computing the ground state energy of the Pöschl-Teller operator $\mathcal{H}_\mu=-\frac{d^2}{ds^2}+d-1+\mu-(p-1)\,u_{\mu,*}^{p-2}$, that is the lowest eigenvalue $\lambda_1(\mu)$ in the eigenvalue problem

$$\mathcal{H}_\mu\,\varphi_1=\lambda_1(\mu)\,\varphi_1\,.$$

See [18, 19] for further references. The function

$$\varphi_1(s):=\alpha^{\frac{p}{2}}\,(\cosh(\beta\,s))^{-\frac{p}{p-2}}=u_{\mu,*}^{p/2} \tag{A.2}$$

solves

$$-\varphi_1''+\frac{1}{4}\,\mu\,p^2\,\varphi_1-(p-1)\,u_{\mu,*}^{p-2}\,\varphi_1=0$$

and provides a solution with

$$\lambda_1(\mu)=d-1+\mu-\frac{1}{4}\,\mu\,p^2\,.$$

The Sturm-Liouville theory guarantees that φ_1 generates the ground state and

$$\inf_{\substack{\varphi\in\mathrm{H}^1(\mathbb{R}^d)\\ \|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}=\|u_{\mu,*}\|^p_{\mathrm{L}^p(\mathcal{C})}}}\frac{q[\mu,\varphi]}{\|u_{\mu,*}\|^p_{\mathrm{L}^p(\mathcal{C})}}=\lambda_1(\mu)=\frac{\int_{\mathcal{C}}\varphi_1\,\mathcal{H}_\mu\,\varphi_1\,dy}{\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}}\,.$$

Notice that the condition $\lambda_1(\mu_{\mathrm{FS}})=0$ determines

$$\mu_{\mathrm{FS}}=4\,\frac{d-1}{p^2-4}\,.$$

Moreover, this shows that

$$\lambda_1(\mu) = -\frac{p^2-4}{4}\,(\mu-\mu_{\rm FS})\,.$$

Other eigenvalues of $\mathcal{H}_\mu$ can also be computed using classical transformations and special functions: see [18, p. 74]. Notice that in Section 3, we normalize the function $\varphi = \varphi_1\, f_1$ in the expansion (11) by the condition $\|\varphi\|_{\mathrm{L}^2(\mathcal{C})} = \|u_{\mu,*}\|_{\mathrm{L}^p(\mathcal{C})}$ consistently with (A.2).

Appendix A.3. Useful quantities

Collecting results of Sections Appendix A.1 and Appendix A.2, with $\alpha^{p-2} = \frac{p}{2}\,\mu$, we find that

$$\frac{\int_{\mathcal{C}} \varphi^2\,dy}{\int_{\mathcal{C}} u_{\mu,*}^p\,dy} = 1\,,$$

$$\frac{\int_{\mathcal{C}} u_{\mu,*}^{p-2}\,\varphi^2\,dy}{\int_{\mathcal{C}} u_{\mu,*}^p\,dy} = \frac{f(q+2)}{f(q)}_{|q=\frac{2p}{p-2}}\,\alpha^{p-2} = \frac{p^2\,\mu}{3\,p-2}\,,$$

$$\frac{\int_{\mathcal{C}} u_{\mu,*}^{p-4}\,\varphi^4\,dy}{\int_{\mathcal{C}} u_{\mu,*}^p\,dy} = \frac{f(q+4)}{f(q)}_{|q=\frac{2p}{p-2}}\,\alpha^{2\,(p-2)} = \frac{2\,p^3\,(p-1)\,\mu^2}{(3\,p-2)\,(5\,p-6)}\,,$$

and

$$\begin{aligned}\frac{\int_{\mathcal{C}} |\nabla\varphi|^2\,dy}{\int_{\mathcal{C}} u_{\mu,*}^2\,dy} &= \alpha^{p-2}\,\frac{\mathsf{I}_p}{\mathsf{I}_2}\left[(d-1)+\left(\frac{p}{p-2}\right)^2\beta^2\left(1-\frac{f(q+2)}{f(q)}\right)_{|q=\frac{2p}{p-2}}\right]\\ &= \frac{1}{2}\,\frac{p\,\mu}{p+2}\left[4\,(d-1)+\frac{p^2\,(p-2)\,\mu}{3\,p-2}\right].\end{aligned}$$

If $\mu=\mu_{\rm FS}$, then we find that $\frac{\int_{\mathcal{C}} |\nabla\varphi|^2\,dy}{\int_{\mathcal{C}} u_{\mu,*}^2\,dy} = \frac{2\,p\,(p-2)\,(p^2+p-1)}{(p+2)\,(3\,p-2)}\,\mu_{\rm FS}^2$.

Appendix A.4. First spherical harmonics

Denote by $\zeta\in[0,\pi]$ the azimuthal angle and consider the Laplace-Beltrami operator $\mathcal{L}$ on the sphere $\mathbb{S}^{d-1}$. When $\mathcal{L}$ is restricted to functions on $\mathbb{S}^{d-1}$ depending only on ζ, it takes the form

$$\mathcal{L}\,f = \sin^{2-d}\zeta\,\frac{d}{d\zeta}\left(\sin^{d-2}\zeta\,\frac{df}{d\zeta}\right)$$

and $\mathcal{L}$ is unitarily equivalent to L defined by

$$\mathsf{L}\,g = (1-x^2)\,g'' - (d-1)\,x\,g' \qquad x\in[-1,1]$$

whose eigenvalues are the *Gegenbauer polynomials* or *ultra-spherical polynomials.* The correspondence between the operators is simply given by

$$f(\zeta) = g(\cos\zeta)$$

and one can check that

$$\int_0^\pi |f(\zeta)|^2 \sin^{d-2}\zeta \, d\zeta = \int_{-1}^1 |g(x)|^2 \, (1-x^2)^{\frac{d-2}{2}} \, dx \, .$$

It is also not difficult to check that the first Gegenbauer polynomials are

$$g_0(x) = 1 \, , \quad g_1(x) = x \, , \quad g_2(x) = d\,x^2 - 1 \, , \quad g_3(x) = (d+2)\,x^3 - 3\,x \, ,$$

with eigenvalues respectively equal to $\Lambda_0 = 0$, $\Lambda_1 = d-1$, $\Lambda_2 = 2\,d$ and $\Lambda_3 = 3\,(d+1)$:

$$-\,\mathsf{L}\,g_0 = 0 \, , \, -\,\mathsf{L}\,g_1 = (d-1)\,g_1 \, , \, -\,\mathsf{L}\,g_2 = 2\,d\,g_2 \, , \, -\,\mathsf{L}\,g_3 = 3\,(d+1)\,g_3 \, .$$

On $[0,\pi]$, we consider the probability measure

$$d\nu(\zeta) = \frac{1}{Z_d} \, \sin^{d-2}\zeta \, d\zeta$$

where

$$Z_d = \int_0^\pi \sin^{d-2}\zeta \, d\zeta = \sqrt{\pi} \, \frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \, .$$

Then

$$f_0(\zeta) = 1 \, , \; f_1(\zeta) = \sqrt{d} \cos\zeta \, , \; f_2(\zeta) = \sqrt{\frac{d+2}{2\,(d-1)}} \, (d\,\cos^2\zeta - 1)$$

are normalized eigenfunctions in $L^2([0,\pi], d\nu(\zeta))$:

$$\int_0^\pi |f_i|^2 \, d\nu = 1 \quad \forall\, i = 0\,, \; 1\,, \; 2\,,$$

with eigenvalues $\Lambda_0 = 0$, $\Lambda_1 = d-1$ and $\Lambda_2 = 2\,d$:

$$-\,\mathcal{L}\,f_0 = 0 \, , \; -\,\mathcal{L}\,f_1 = (d-1)\,f_1 \, , \; -\,\mathcal{L}\,f_2 = 2\,d\,f_2 \, .$$

We also have the useful formulae

$$\int_0^\pi |f_1|^4 \, d\nu = \frac{3\,d}{d+2} \, , \quad \kappa_{(d)} := \int_0^\pi |f_1|^2 \, f_2 \, d\nu = \sqrt{\frac{2\,(d-1)}{d+2}}$$

and

$$f_1^2 = f_0 + \sqrt{\frac{2\,(d-1)}{d+2}} \, f_2 = f_0 + \kappa_{(d)} \, f_2 \, .$$

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